Representations for complex numbers with integer digits

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Real expansions

Beta-expansions

For a real β with $|\beta| > 1$ and $\varepsilon \in [0,1)$ we define the *beta-transformation* (with respect to the base β and the interval $[-\varepsilon, 1-\varepsilon)$) by

$$T_{\beta,\varepsilon} : [-\varepsilon, 1-\varepsilon) \longrightarrow [-\varepsilon, 1-\varepsilon), x \longmapsto \beta x \pmod 1$$
$$= \beta x - \lfloor \beta x + \varepsilon \rfloor$$

For $x \in [-\varepsilon, 1-\varepsilon)$ we let $\boldsymbol{e}_{\beta,\varepsilon}(x) := (e_n)_{n\geq 1}$ with $e_n = \beta T^{n-1}(x) - T^n(x) \in \mathbb{N}$ denote the digit sequence induced by the successive application of $T_{\beta,\varepsilon}$ on x. We have $\boldsymbol{e}_{\beta,\varepsilon}(x) \in \mathcal{N}_{\beta,\varepsilon}$ with

$$\mathcal{N}_{\beta,\varepsilon} := \begin{cases} \{e \in \mathbb{Z} : -\beta\varepsilon \leq e < (1-\varepsilon)\beta\} & \text{if } \beta > 1, \\ \{e \in \mathbb{Z} : (1-\varepsilon)\beta < e \leq -\beta\varepsilon\} & \text{if } \beta < -1. \end{cases}$$

The beta-expansion of x (with respect to (β,ε)) is the representation

$$x = \frac{e_1}{\beta} + \frac{e_2}{\beta^2} + \frac{e_3}{\beta^3} + \cdots \qquad \text{(beta-expansion)}.$$

We call an integer sequence (β,ε)-admissible if it is produced by successive application of $T_{\beta,\varepsilon}$ on some $x \in [-\varepsilon, 1-\varepsilon)$.

Prominent settings

- for $\beta = q \in \mathbb{N}$, $\varepsilon = 0$ we obtain the q-ary expansions;
- for $\beta = 3$, $\varepsilon = \frac{1}{2}$ we obtain the balanced ternary expansion (Knuth);
- for $\beta > 1$, $\varepsilon = 0$ we obtain the classical beta-expansion (Rényi, Parry);
- for $\beta > 1$, $\varepsilon = \frac{1}{2}$ we obtain the symmetric beta-expansion (Akiyama-Scheicher);
- for $\beta < 1$, $\varepsilon = \frac{\beta}{1-\beta}$ we obtain the $(-\beta)$-expansion (Ito-Sadahiro);
- ...

Complex expansions

Canonical number systems (cf. Kátai, Kovács, Kőrnyei, Pethő, ...)

Suppose that ζ is an algebraic integers that induces a Canonical number system.

Then each complex number z can be represented with respect to ζ and digit set

$$\{0, 1, \ldots, |\zeta| - 1\}.$$

Problems: the concept does not consider non-algebraic bases ζ,

It is difficult to obtain the representation for an (arbitrary) $z \in \mathbb{C}$.

Complex expansions

Rotational beta-expansions (Akiyama-Caalim - 2017)

Let $\zeta = \beta \cdot \xi$ such that $\beta > 1$ is a real number and $\xi \in \mathbb{C} \setminus \mathbb{R}$ satisfies $|\xi| = 1$.
Furthermore, fix $\theta, \eta_1, \eta_2 \in \mathbb{C}$ such that $\eta_1/\eta_2 \notin \mathbb{R}$. Then
$D := \{\theta + \mu_0 \cdot \eta_1 + \mu_1 \cdot \eta_2 : \mu_0, \mu_1 \in [0,1)\}$ is a fundamental domain of the lattice $\mathfrak{L}$ generated by η_1 and η_2.
The rotational beta-transformation is defined by

$$S : D \longrightarrow D, z \longmapsto \beta\xi \cdot z \pmod{\mathfrak{L}}.$$

For each $z \in D$ the *rotational beta-expansion* (with respect to S) is the representation

$$z = \frac{d_1}{\beta\xi} + \frac{d_2}{(\beta\xi)^2} + \frac{d_3}{(\beta\xi)^3} + \cdots$$

where $d_n = \beta\xi S^{n-1}(z) - S^n(z) \in \mathfrak{L}$ for each $n \geq 1$.

We are interested in rotational beta-expansions where $\eta_1 := -\overline{\zeta}$, $\eta_2 := 1$ and $\theta := -\varepsilon(\eta_1 + \eta_2)$ for an $\varepsilon \in [0,1)$.

The zeta-expansion

Definition

Zeta-expansions

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$, $\varepsilon \in [0, 1)$ and

$$D_{\zeta,\varepsilon} := \{\mu_0(-\overline{\zeta}) + \mu_1 : \mu_0, \mu_1 \in [-\varepsilon, 1-\varepsilon).$$

We define the *zeta-transformation* (with respect to the pair (ζ, ε)) by

$$S_{\zeta,\varepsilon} : D_{\zeta,\varepsilon} \longrightarrow D_{\zeta,\varepsilon}, z \longmapsto \zeta \cdot z \pmod{\mathfrak{L}}$$

where $\mathfrak{L}$ is the lattice generated by $-\overline{\zeta}$ and 1.

For an $z \in D_{\zeta,\varepsilon}$ we define the digit string produced by the successive application of $S_{\zeta,\varepsilon}$ by

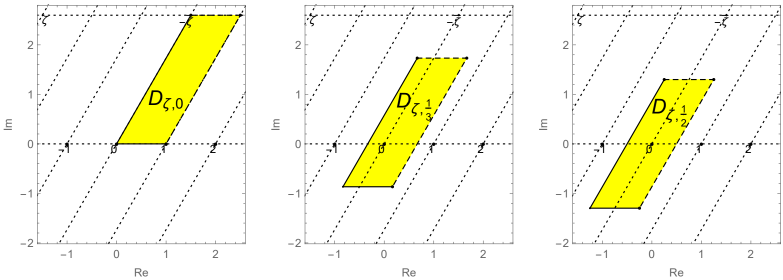
$$\boldsymbol{d}_{\zeta,\varepsilon}(z) = \left(\zeta S^{n-1}(z) - S^n(z)\right)_{n\geq 1} \in \mathfrak{L}^{\mathbb{N}}.$$

If $|\zeta| > 1$ then the *zeta-expansion* of z (with respect to $S_{\zeta,\varepsilon}$) is the representation

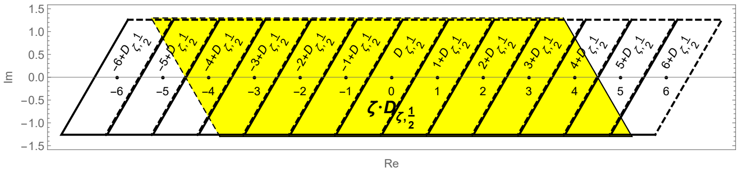
$$x = \frac{d_1}{\zeta} + \frac{d_2}{\zeta^2} + \frac{d_3}{\zeta^3} + \cdots$$

where $\boldsymbol{d}_{\zeta,\varepsilon}(z) = (d_n)_{n\geq 1}$.
We call an integer sequence (ζ, ε)-admissible if it is produced by successive application of $S_{\beta,\varepsilon}$ on some $x \in [-\varepsilon, 1-\varepsilon)$.

Example $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$



The fundamental domain $D_{\zeta,\varepsilon}$ for different choices of ε.
Left: $\varepsilon = 1$, centre: $\varepsilon = 1/3$, right: $\varepsilon = 1/2$



The zeta-transformation induces integer digits: Integer translates of the fundamental domain $D_{\zeta,1/2}$ completely cover $\zeta \cdot D_{\zeta,1/2}$.

Properties

We consider the complex plane as two dimensional vector space over $\mathbb{R}$. We can uniquely represent each $z \in \mathbb{C}$ with respect to the base $\{-\overline{\zeta}, 1\}$. Let

$$\psi_\zeta : \mathbb{C} \longrightarrow \mathbb{R}^2, z \longmapsto (\mu_0, \mu_1) = \left(\frac{z - \overline{z}}{\zeta - \overline{\zeta}}, \frac{z \cdot \zeta - \overline{z \cdot \zeta}}{\zeta - \overline{\zeta}} \right).$$

Then $z = -\overline{\zeta}\mu_0 + \mu_1$ and $z \in D_{\zeta,\varepsilon} \Longleftrightarrow \psi_\zeta(z) \in [-\varepsilon, 1 - \varepsilon)^2$.

Lemma

Let $a_0 := -\zeta\overline{\zeta}$ and $a_1 := \zeta + \overline{\zeta}$ and define

$$\boldsymbol{A}_{a_0,a_1} : [-\varepsilon, 1 - \varepsilon)^2 \longrightarrow [-\varepsilon, 1 - \varepsilon)^2, (\mu_0, \mu_1) \longmapsto (\mu_1, a_0\mu_0 + a_1\mu_1) \pmod{\mathbb{Z}^2}.$$

Then $\psi_\zeta \circ S_{\zeta,\varepsilon}(z) = S_{\zeta,\varepsilon} \circ \psi_\zeta(z)$ for all $z \in D_{\zeta,\varepsilon}$.

The map $\boldsymbol{A}_{a_0,a_1}$ corresponds to a piecewise affine map of the torus. For the case $|\zeta| = 1$ this maps has interesting dynamical properties and applications in signal processing (second order digital filters).

Properties

We obtain that

$$\psi_\zeta \circ S_{\zeta,\varepsilon}(z) = (\mu_1, -\zeta\overline{\zeta}\cdot\mu_0 + (\zeta+\overline{\zeta})\mu_1 - d) \text{ with } d = \left\lfloor -\zeta\overline{\zeta}\cdot\mu_1 + (\zeta+\overline{\zeta})\mu_2 + \varepsilon \right\rfloor.$$

Proposition

Let $\zeta \in \mathbb{C}\setminus\mathbb{R}$ and $\varepsilon \in [0,1)$. For each $z \in D_{\zeta,\varepsilon}$ we have

$$S_{\zeta,\varepsilon}(z) = \zeta\cdot z - \left\lfloor \frac{z\cdot\zeta^2 - \overline{z\cdot\zeta^2}}{\zeta-\overline{\zeta}} + \varepsilon \right\rfloor.$$

Therefore, $\boldsymbol{d}_{\zeta,\varepsilon}(z)$ is a sequence of integers that are contained in the set of digits

$$\mathcal{N}_{\zeta,\varepsilon} = \left\{ d \in \mathbb{Z} : (\varepsilon-1)|\zeta-1|^2 < d \le \varepsilon|\zeta-1|^2 \right\} \quad \text{if } \mathrm{Re}(\zeta) \le 0,$$

$$\mathcal{N}_{\zeta,\varepsilon} = \left\{ d \in \mathbb{Z} : (\varepsilon-1)|\zeta-1|^2 - 2\mathrm{Re}(\zeta) < d < \varepsilon|\zeta-1|^2 + 2\mathrm{Re}(\zeta) \right\} \quad \text{if } \mathrm{Re}(\zeta) > 0.$$

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

$$a_0 = -\zeta\overline{\zeta} = -|\zeta|^2 = -9 \qquad a_1 = \zeta + \overline{\zeta} = \mathrm{Re}(\zeta) = -3$$

$$(t-\zeta)(t-\overline{\zeta}) = t^2 - a_1 t - a_0 \qquad |\zeta - 1|^2 = 1 - a_1 - a_0 = 13$$

$$\mathcal{N}_{\zeta,\varepsilon} = \{-6, -5, \dots, 5, 6\}$$

We consider $z = 1/4$, hence $\psi(z) = (0, 1/4)$.

$$\begin{array}{lll}
\psi_\zeta(z) = (0, 1/4) & \longmapsto (1/4, -9\cdot 0 - 3/4 - \lfloor -9\cdot 0 - 3/4 + 1/2 \rfloor) & \\
 & = (1/4, -3/4 - \lfloor -1/4 \rfloor) = (1/4, 1/4) & d_1 = -1 \\
\psi_\zeta(S_{\zeta,\varepsilon}(z)) = (1/4, 1/4) & \longmapsto (1/4, -9/4 - 3/4 - \lfloor -9/4 - 3/4 + 1/2 \rfloor) & \\
 & = (1/4, -3 - \lfloor 5/2 \rfloor) = (1/4, 0) & d_2 = -3 \\
\psi_\zeta(S^2_{\zeta,\varepsilon}(z)) = (1/4, 0) & \longmapsto (0, -9/4 - \lfloor -7/4 \rfloor) = (0, -1/4) & d_3 = -2 \\
\psi_\zeta(S^3_{\zeta,\varepsilon}(z)) = (0, -1/4) & \longmapsto (-1/4, 3/4 - \lfloor 5/4 \rfloor) = (-1/4, -1/4) & d_4 = 1 \\
\psi_\zeta(S^4_{\zeta,\varepsilon}(z)) = (-1/4, -1/4) & \longmapsto (-1/4, 3 - \lfloor 7/2 \rfloor) = (-1/4, 0) & d_3 = 3 \\
\psi_\zeta(S^5_{\zeta,\varepsilon}(z)) = (-1/4, 0) & \longmapsto (0, 9/4 - \lfloor 11/4 \rfloor) = (0, 1/4) & d_3 = 2
\end{array}$$

We see that $\boldsymbol{d}_{\zeta,\varepsilon}(z) = (-1, -3, -2, 1, 3, 2)^\omega$ and

$$z = \frac{1}{4} = \frac{-1}{\zeta} + \frac{-3}{\zeta^2} + \frac{-2}{\zeta^3} + \frac{1}{\zeta^4} + \frac{3}{\zeta^5} + \frac{2}{\zeta^6} + \frac{-1}{\zeta^7} + \cdots$$

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

z	$\boldsymbol{\psi}_\zeta(z)$	$\boldsymbol{d}_{\zeta,\varepsilon}(z)$
$\frac{1}{4}$	$(0, 1/4)$	$(-1, -3, -2, 1, 3, 2)^\omega$
$-\frac{1}{4}$	$(0, -1/4)$	$(1, 3, 2, -1, -3, -2)^\omega$
$\frac{1}{3}$	$(0, 1/3)$	$-1, -3, (0)^\omega$
$-\frac{1}{2} - \frac{\sqrt{3}i}{2}$	$(-1/3, 0)$	$3, (0)^\omega$
$\frac{1}{6} - \frac{\sqrt{3}i}{6}$	$(-1/9, 1/3)$	$0, -3, (0)^\omega$
$-\frac{5}{26} - \frac{5\sqrt{3}i}{26}$	$(-1/13, -1/13)$	$(1)^\omega$
$-\frac{1}{14} - \frac{5\sqrt{3}i}{14}$	$(-1/7, 1/7)$	$(1, -1)^\omega$
$-\frac{1}{2}$	$(0, -1/2)$	$(2, 6, 5)^\omega$
$-\frac{3}{4} - \frac{3\sqrt{3}i}{4}$	$(-1/2, 0)$	$(5, 2, 6)^\omega$
$-\frac{5}{4} - \frac{3\sqrt{3}i}{4}$	$(-1/2, -1/2)$	$(6, 5, 2)^\omega$
$\frac{1}{24} + \frac{3\sqrt{3}i}{8}$	$(1/4, -1/3)$	$-1, 4, (3, 2, -1, -3, -2, 1)^\omega$

Admissible sequences and soficness

Admissible sequences for the beta-transformation

Proposition

Let $\beta \in \mathbb{R}$ with $|\beta| > 1$ and $\varepsilon \in [0,1)$. An integer sequence $(e_n)_{n\geq 1}$ is (β,ε)-admissible if and only if for all $m \geq 0$ we have

$$\sum_{n\geq 1} e_{n+m}\beta^{-n} \in [-\varepsilon, 1-\varepsilon).$$

Admissible sequences for the zeta-transformation

For each $n \in \mathbb{Z}$ let

$$P_n(\zeta) := \frac{\zeta^n - \overline{\zeta}^n}{\zeta - \overline{\zeta}}.$$

We have $P_n(\zeta) = |\zeta|^{2n} P_{-n}(\zeta)$ and $P_n(\zeta) = 2\mathrm{Re}\,(\zeta)\, P_{n-1}(\zeta) - |\zeta|^2 P_{n-2}(\zeta)$.
For $n \geq 1$:

$$P_n(\zeta) = E_n\big(2\mathrm{Re}\,(\zeta)\,, |\zeta|^2\big) = |\zeta| U_n\big(\mathrm{Re}\,(\zeta) \cdot |\zeta|^{-1}\big),$$

where E_n is the nth Dickson polynomial of the second kind and U_n is the nth Chebyshev polynomial of the second kind.

Proposition

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and $\varepsilon \in [0, 1)$. An integer sequence $(d_n)_{n\geq 1}$ is (ζ, ε)-admissible if and only if for all $m \geq 0$ we have

$$\sum_{n\geq 1} d_{n+m} \zeta^{-n} \in D_{\zeta,\varepsilon}$$

which is in turn equivalent to the fact that for all $m \geq 0$ we have

$$\sum_{n\geq 1} d_{n+m} P_{-n}(\zeta) \in [-\varepsilon, 1 - \varepsilon).$$

Soficness of the underlying shift

Proposition

Let $\beta \in \mathbb{R}$ be a Pisot number and $\varepsilon \in [0,1) \cap \mathbb{Q}(\beta)$. Then the shift space (beta-shift)

$$\Omega_{\beta,\varepsilon} := \overline{\{\boldsymbol{e}_{\beta,\varepsilon}(x) : x \in [-\varepsilon, 1-\varepsilon)\}}$$

is sofic.

Proposition (Akiyama-Caalim - 2017)

Let $\zeta = \beta\xi \in \mathbb{C} \setminus \mathbb{R}$ such that β is a Pisot number and ξ is a root of unity. Furthermore, suppose that $\mathrm{Re}\,(\zeta) \in \mathbb{Q}(\beta)$ and $\varepsilon \in [0,1) \cap \mathbb{Q}(\beta)$. Then the shift space (zeta-shift)

$$\Omega_{\zeta,\varepsilon} := \overline{\{\boldsymbol{d}_{\beta,\varepsilon}(z) : z \in D_{\zeta,\varepsilon}\}}$$

is sofic.

Extension to the entire complex plane

Extension to the entire real line

Theorem

Let $\beta \in \mathbb{R}$ with $\beta > 1$ and $\varepsilon \in (0,1)$ and suppose that

$$\beta^{-1} \cdot [-\varepsilon, 1-\varepsilon) \subset [-\varepsilon, 1-\varepsilon).$$

Then for each real number $x \in \mathbb{R} \setminus \{0\}$ there exists an integer m and a (β, ε)-admissible sequence $(e_n)_{n \geq 1}$, both uniquely determined, that satisfy $e_1 \neq 0$ and

$$x = \sum_{n \geq 1} e_n \beta^{-n+m}.$$

We write

$$(x)_{\zeta,\varepsilon} = \begin{cases} 0\bullet\underbrace{0\cdots0}_{m}\,e_1 e_2 \cdots & \text{if } m \geq 0, \\ e_1 e_2 \cdots e_{-m}\bullet e_{-m+1} e_{-m+2} \cdots & \text{if } m < 0. \end{cases}$$

For $x = 0$ we write $(0)_{\zeta,\varepsilon} = 0 \bullet \dot{0}$.

Extension to the entire complex plane

We want to extend our representation to the entire complex plane. We require $\varepsilon \neq 0$ and

$$\zeta^{-1} D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}.$$

Theorem

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| > 1$ and $\varepsilon \in [0,1)$ such that $\zeta^{-1} D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}$. Then for each complex number $z \in \mathbb{C} \setminus \{0\}$ there exists an integer m and a (ζ,ε)-admissible sequence $(d_n)_{n\geq 1}$, both uniquely determined, that satisfy $d_1 \neq 0$ and

$$z = \sum_{n\geq 1} d_n \zeta^{-n+m}.$$

We write

$$(z)_{\zeta,\varepsilon} = \begin{cases} 0\bullet\underbrace{0\cdots 0}_{m} d_1 d_2 \cdots & \text{if } m \geq 0, \\ d_1 d_2 \cdots d_{-m}\bullet d_{-m+1} d_{-m+2} \cdots & \text{if } m < 0. \end{cases}$$

For $z = 0$ we write $(0)_{\zeta,\varepsilon} = 0 \bullet \dot{0}$.

We have $z \in D$ if and only if the zeta-expansion of z has no integer part, i.e. it starts with $0\bullet$.

Settings that satisfy $D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}$

Proposition (cf. Dombek-Masáková-Pelantová - 2011)

For $\beta < -1$ we have $\beta^{-1} \cdot [-\varepsilon, 1-\varepsilon) \subset [-\varepsilon, 1-\varepsilon)$ if and only if

$$\varepsilon \in \left[\frac{1}{\beta+1}, 1 - \frac{\beta}{\beta+1}\right)$$

Proposition

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| > 1$. Then $\zeta \cdot D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}$ if and only if

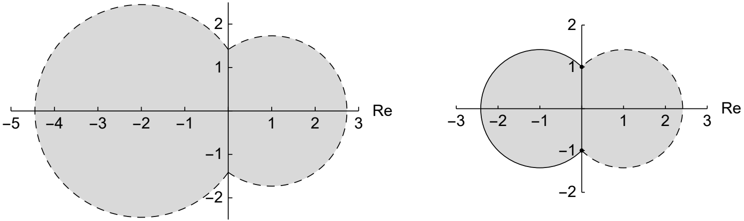
$$\varepsilon \in \begin{cases} \left[\frac{1}{|\zeta-1|^2}, 1 - \frac{1}{|\zeta-1|^2}\right] & \text{for } \mathrm{Re}\,(\zeta) > 0, \\ \left[1 - \frac{|\zeta|^2}{|\zeta-1|^2}, \frac{|\zeta|^2}{|\zeta-1|^2}\right) & \text{for } \mathrm{Re}\,(\zeta) \leq 0. \end{cases}$$

Settings that satisfy $D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}$

Corollary

Let $\varepsilon \in (0,1)$. Then $\zeta \cdot D_{\zeta,\varepsilon} \subset D_{\zeta,\varepsilon}$ if and only if

$$\begin{array}{lll} |\zeta - 1|^2 \geq \varepsilon^{-1} & \wedge |\zeta + (1-\varepsilon)\varepsilon^{-1}|^2 \geq (1-\varepsilon)\varepsilon^{-2} & \text{if } \varepsilon \in (0, 1/2), \\ |\zeta - 1|^2 \geq (1-\varepsilon)^{-1} & \wedge |\zeta + \varepsilon(1-\varepsilon)^{-1}|^2 > \varepsilon(1-\varepsilon)^{-2} & \text{if } \varepsilon \in [1/2, 1). \end{array}$$



Grey: the set of “forbidden” bases ζ for $\varepsilon = 1/3$ (left) and $\varepsilon = 1/2$ (right)

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

z	$\psi_\zeta(z)$	$(z)_{\zeta,\varepsilon}$
$\frac{1}{4}$	$(0, {}^1/_4)$	$0 \bullet \overline{(-1)(-3)(-2)132}$
$\frac{1}{3}$	$(0, {}^1/_3)$	$0 \bullet (-1)(-3)\dot{0}$
$-\frac{3}{2} + \frac{3\sqrt{3}i}{2}$	$(1, -3)$	$10 \bullet \dot{0}$
$-\frac{3}{2} - \frac{3\sqrt{3}i}{2}$	$(-1, 0)$	$0 \bullet (-1)(-3)\dot{0}$
20	$(0, 20)$	$1132 \bullet \dot{0}$
2020	$(0, 2020)$	$3133134 \bullet \dot{0}$
2000	$(0, 2000)$	$3132002 \bullet \dot{0}$
$-\frac{12}{7} + \frac{81\sqrt{3}i}{14}$	$({}^{27}/_7, -{}^{15}/_7)$	$45 \bullet \overline{35204(-1)}$

Relations between complex and real expansions

Relations between real expansions with integer base

Proposition

Let $\beta = N \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $\varepsilon \in [0, 1)$. A sequence $(e_n)_{n\geq 1}$ of bounded integers is (N, ε)-admissible if and only if $(Ne_{2n-1} + e_{2n})_{n\geq 1}$ and $(Ne_{2n} + e_{2n+1})_{n\geq 1}$ are (N^2, ε)-admissible.

Example: The binary system ($(2, 0)$-expansion) can easily be transferred into the hexadecimal system ($(16, 0)$-expansion) by joining digits.

$$
\begin{aligned}
(\pi)_{2,0} &= \underbrace{0011}\bullet\underbrace{0010}\underbrace{0100}\underbrace{0011}\underbrace{1111}\underbrace{0110}\underbrace{1010} \quad \ldots \\
(\pi)_{16,0} &= \;3 \;\bullet\; 2 \quad 4 \quad 3 \quad F \quad 6 \quad A \quad \ldots
\end{aligned}
$$

Multiples of fourth roots of unity

If $\zeta = \beta \cdot i$ is an imaginary base ($\beta \in \mathbb{R}$) and $z = -\overline{\zeta}\mu_0 + \mu_1$ then we can study the imaginary part $-\beta\mu_0$ and the real part μ_1 separately:

$$\begin{array}{rcccccccc} \boldsymbol{e}_{-\beta^2,\epsilon}(\mu_0) = & d_1, & & d_3, & & d_5, & & & \cdots \\ \boldsymbol{d}_{\zeta,\epsilon}(-\overline{\zeta}\mu_0 + \mu_1) = & d_1, & d_2, & d_3, & d_4, & d_5, & d_6, & & \cdots \\ \boldsymbol{e}_{-\beta^2,\epsilon}(\mu_1) = & & d_2, & & d_4, & & d_6, & & \cdots \end{array}$$

Theorem

Let $\beta > 1$ be a real number, $\zeta = \pm\beta i$, $\varepsilon \in [0,1)$ and $(d_n)_{n\geq 1}$ an integer sequence. Then the following assertions are equivalent.

- The sequence $(d_n)_{n\geq 1}$ is (ζ, ε)-admissible.
- The sequences $(d_{2n-1})_{n\geq 1}$ as well as $(d_{2n})_{n\geq 1}$ are $(-\beta^2, \varepsilon)$-admissible.

Multiples of third and sixth roots of unity

Theorem

Let $N \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $\zeta = Ne^{\pm 2\pi i/3}$, $\varepsilon \in [0, 1)$, and $(d_n)_{n\geq 1}$ be a sequence of bounded integers. Then the following assertions are equivalent.

- The sequence $(d_n)_{n\geq 1}$ is (ζ, ε)-admissible.
- For each $k \in \{0, 1, 2\}$ the sequence $(e_n^{(k)})_{n\geq 1}$ is (N^3, ε)-admissible, where $e_n^{(k)} := -Nd_{3n-2+k} + d_{3n-1+k}$.

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

We have $\zeta = 3 \cdot e^{2\pi/3}$, therefore the complex zeta-transformation $S_{\zeta,\varepsilon}$ is related with the real beta-transformation $T_{27,\varepsilon}$.

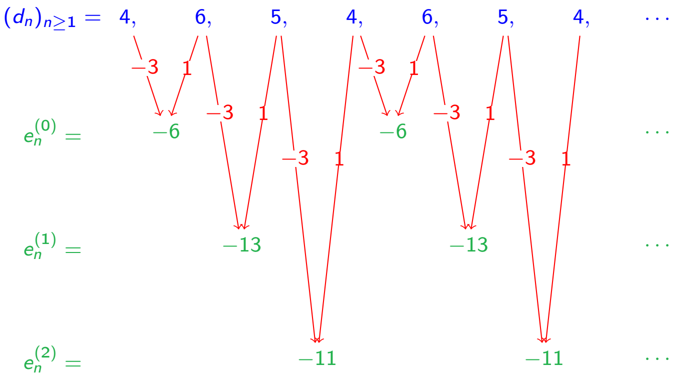
We have $\boldsymbol{e}_{27,1/2}(-\frac{1}{2}) = (-13)^{\omega}$. Therefore, an integer sequence $(e_n)_{n\geq 1}$ is $(27, -1/2)$-admissible if and only if

$$\forall m \geq 1 : \quad (-13)^{\omega} \leq_{\text{lex}} (e_n)_{n\geq m} <_{\text{lex}} (13)^{\omega}.$$

An integer sequence $(d_n)_{n\geq 1}$ is $(\zeta, 1/2)$-admissible if and only if

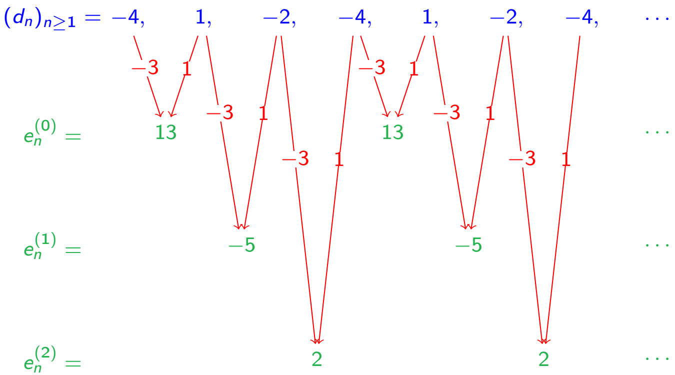
$$\begin{aligned}
\forall m \geq 1 : &\quad (-13)^{\omega} \leq_{\text{lex}} (-3d_{3n-2} + d_{3n-1})_{n\geq m} <_{\text{lex}} (13)^{\omega} \\
\forall m \geq 1 : &\quad (-13)^{\omega} \leq_{\text{lex}} (-3d_{3n-1} + d_{3n})_{n\geq m} <_{\text{lex}} (13)^{\omega} \\
\forall m \geq 1 : &\quad (-13)^{\omega} \leq_{\text{lex}} (-3d_{3n} + d_{3n+1})_{n\geq m} <_{\text{lex}} (13)^{\omega}
\end{aligned}$$

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$



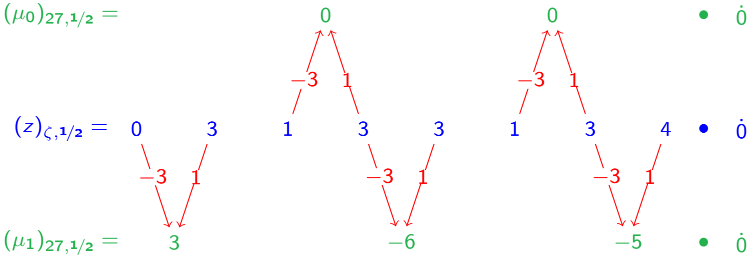
$\Longrightarrow (4, 6, 5)^\omega$ is $(\zeta, 1/2)$-admissible

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$



$\Longrightarrow (-4, 1, -2)^{\omega}$ is not $(\zeta, 1/2)$-admissible

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

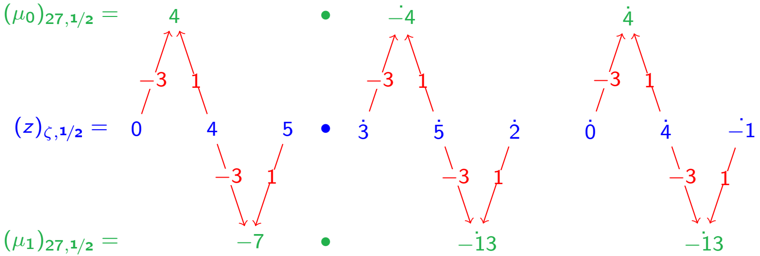


$$\mu_0 = 0$$
$$\mu_1 = 3 \cdot 27^2 - 6 \cdot 27 - 5 = 2020$$

$$z = ?$$

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

$$z = -\frac{12}{7} + \frac{81\sqrt{3}i}{14} = -\bar{\zeta} \cdot \frac{27}{7} - \frac{15}{7} \Longrightarrow \begin{array}{l} \mu_0 = \frac{27}{7} \\ \mu_1 = -\frac{15}{7} \end{array}$$



Eighth roots of unity

Theorem

Let $N \in \mathbb{Z} \setminus \{0\}$, $\zeta = \sqrt{2}Ne^{\pm\pi i/4}$, $\varepsilon \in [0,1)$, and $(d_n)_{n\geq 1}$ be a sequence of bounded integers.. Then the following assertions are equivalent.

- The sequence $(d_n)_{n\geq 1}$ is (ζ,ε)-admissible.
- For each $k \in \{0,1,2,3\}$ the sequence $(e_n^{(k)})_{n\geq 1}$ is $(-4N^4,\varepsilon)$-admissible, where $e_n^{(k)} := 2N^2 d_{4n-3+k} + 2Nd_{4n-2+k} + d_{4n-1+k}$.

Zeta-expansions and Canonical number systems

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

Theorem (e.g. Kátai-Kőrnyei - 1992)

Each complex number z can be represented as

$$z = \sum_{n \geq 1} d_n \zeta^{-n+m}, \qquad d_n \in \{-4, \dots, 4\}.$$

The representation is (up to leading zeros) unique for almost all $z \in \mathbb{C}$.

The set

$$C := \left\{ \sum_{n \geq 1} d_n \zeta^{-n} : (d_n)_{n \geq 1} \in \{-4, -3, \dots, 4\}^{\mathbb{N}} \right\}$$

of numbers with zero integer part is self-similar and satisfies the iterated function system

$$C = \bigcup_{d \in \{-4, \dots, 4\}} \zeta^{-1}(C + d)$$

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$

Theorem

Each complex number z can be represented as

$$z = \sum_{n\geq 1} d_n \zeta^{-n+m}, \qquad d_n \in \{-6, \ldots, 6\}, -13 \leq -3d_n + d_{n+1} \leq 13.$$

The representation is (up to leading zeros) unique for almost all $z \in \mathbb{C}$. We obtain complete uniqueness by requiring that that $(-3d_{3n-k} + d_{3n-k+1})_{n\geq \ell} <_{\text{lex}} (13)^\omega$ holds for all $\ell \geq 1$, $k \in \{0, 1, 2\}$.

The set

$$\overline{D_{\zeta,\varepsilon}} := \Big\{ \sum_{n\geq 1} d_n \zeta^{-n} : (d_n)_{n\geq 1} \in \{-6, -3, \ldots, 6\}^{\mathbb{N}}$$

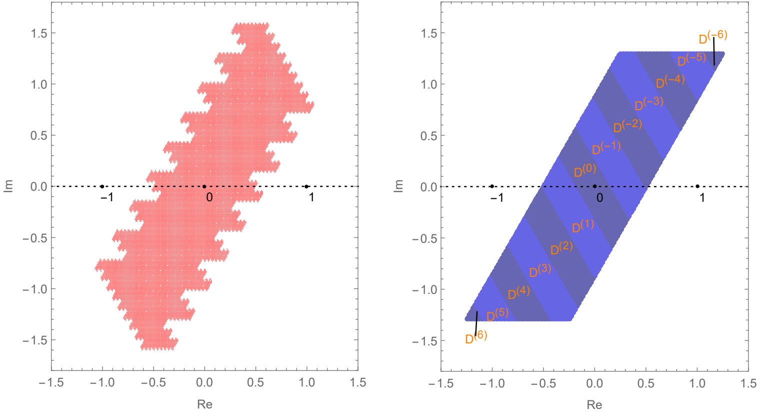
$$\forall n \geq 1 : -13 \leq -3d_n + d_{n+1} \leq 13 \Big\}$$

is the union of the invariant set list $\{D^{(-6)}, \ldots, D^{(6)}\}$ of the graph directed iterated function system

$$D^{(d)} = \bigcup_{\substack{-6 \leq -3d+d' \leq 6 \\ d' \in \mathcal{N}_{\zeta,\varepsilon}}} \zeta^{-1} \cdot \left(d + D^{(d')}\right) \qquad (d \in \mathcal{N}_{\zeta,\varepsilon}).$$

We have $\sum_{n\geq 1} d_n \zeta^{-n} \in \partial(D_{\zeta,\varepsilon})$ if and only if $(-3d_{3n-2} + d_{3n-1})_{n\geq 1} = (\pm 13)^\omega$ or $(-3d_{3n-1} + d_{3n})_{n\geq 1} = (\pm 13)^\omega$.

Example: $\zeta = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$, $\varepsilon = \frac{1}{2}$



The fundamental domains.
Left: the fundamental domains for the Canonical number system; Right: the fundamental domains for the zeta-expansion

Periodicity and Finiteness properties

Definitions

Definition

Let $\beta > 1$ be a real algebraic integer and $\varepsilon \in [0, 1)$. For the pair (β, ε) we define the periodicity property (P-beta) and the finiteness property (F-beta) by

$$\forall x \in [-\varepsilon, 1-\varepsilon) \cap \mathbb{Q}(\beta) : \left\{ T^n_{\beta,\varepsilon}(x) : n \in \mathbb{N} \right\} \text{ is finite} \quad \text{(P-beta)}$$

$$\forall x \in [-\varepsilon, 1-\varepsilon) \cap \mathbb{Z}[\beta^{-1}] : \left\{ T^n_{\beta,\varepsilon}(x) : n \in \mathbb{N} \right\} \text{ contains } 0 \quad \text{(F-beta)}$$

Definition

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic integer and $\varepsilon \in [0, 1)$. For the pair (ζ, ε) we define the periodicity property (P-zeta) and the finiteness property (F-zeta) by

$$\forall z \in D_{\zeta,\varepsilon} \cap \mathbb{Q}(\zeta) : \left\{ S^n_{\zeta,\varepsilon}(z) : n \in \mathbb{N} \right\} \text{ is finite} \quad \text{(P-zeta)}$$

$$\forall z \in D_{\zeta,\varepsilon} \cap \mathbb{Z}[\zeta^{-1}] : \left\{ S^n_{\zeta,\varepsilon}(z) : n \in \mathbb{N} \right\} \text{ contains } 0 \quad \text{(F-zeta)}$$

Pisot and Salem numbers

Definition

An algebraic integer $\beta > 1$ is a *Pisot number* if all algebraic conjugates different from β are located in the open unit disk. We call β a *Salem number* if all Galois conjugates different from β are contained in the closed unit disk where at least one such conjugate is located at the unit circle.

Definition

An algebraic integer $\zeta \in \mathbb{C} \setminus \mathbb{R}$ is a *complex Pisot number* if all algebraic conjugates different from ζ and $\overline{\zeta}$ are located in the open unit disk. We call ζ a *complex Salem number* if all algebraic conjugates different from ζ and $\overline{\zeta}$ are contained in the closed unit disk where at least one such conjugate is located at the unit circle.

The periodicity property

Proposition (Bertrand - 1977, Schmidt - 1980)

Let $\beta\mathbb{R}$ be an algebraic integer and $\varepsilon \in [0,1)$. If β is a Pisot number then (β, ε) satisfies (P-beta). On the other hand if (β, ε) satisfies (P-beta) then β is a Pisot number or Salem number.

Proposition

Let $\zeta \in \mathbb{C}$ be an algebraic integer and $\varepsilon \in [0,1)$. If ζ is a complex Pisot number then (ζ, ε) satisfies (P-zeta). On the other hand if (ζ, ε) satisfies (P-zeta) then ζ is a complex Pisot number or complex Salem number.

Shift radix systems

Definition (Akiyama-Borbély-Brunotte-Pethő-Thuswaldner - 2005)

Let $\varepsilon \in [0,1)$, $\boldsymbol{r} = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$ and define the map $\boldsymbol{\tau}_{\boldsymbol{r},\varepsilon} : \mathbb{Z}^d \longrightarrow \mathbb{Z}^d$ by

$$\begin{aligned} \boldsymbol{\tau}_{\mathbf{r},\varepsilon} : \mathbb{Z}^d &\to \mathbb{Z}^d, \\ (x_0, \ldots, x_{d-1}) &\mapsto \left(x_1, \ldots, x_{d-1}, - \left\lfloor \sum_{j=0}^{d-1} r_j x_j + \varepsilon \right\rfloor \right). \end{aligned}$$

We call the dynamical system $(\mathbb{Z}^d, \boldsymbol{\tau}_{\boldsymbol{r},\varepsilon})$ a *shift radix system* if for each $\boldsymbol{x} \in \mathbb{Z}^d$ the orbit $\{\boldsymbol{\tau}^n_{\boldsymbol{r},\varepsilon}(\boldsymbol{x}) : n \in \mathbb{N}\}$ contains 0.

The finiteness property

Theorem (Akiyama-Borbély-Brunotte-Pethő-Thuswaldner - 2005)

Let $\beta > 1$ be a real algebraic integer with minimal polynomial

$$(x^d + r_{d-1}x^{d-1} + \cdots + r_0)(x - \beta)$$

and $\varepsilon \in [0, 1)$. Then the pair (β, ε) satisfies (F-beta) if and only if $(\mathbb{Z}^d, \tau_{\mathbf{r},\varepsilon})$, with $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$, is a shift radix system.

Theorem

Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ be an algebraic integer with minimal polynomial

$$(x^d + r_{d-1}x^{d-1} + \cdots + r_0)(x - \zeta)(x - \overline{\zeta})$$

and $\varepsilon \in [0, 1)$. Then the pair (ζ, ε) satisfies (F-zeta) if and only if $(\mathbb{Z}^d, \tau_{\mathbf{r},\varepsilon})$, with $\boldsymbol{r} = (r_0, \ldots, r_{d-1})$, is a shift radix system.

The end

Some literature

- Akiyama, Caalim: *Rotational beta expansion: ergodicity and soficness*. J. Math. Soc. Japan 69, 397–415 (2017)
- Akiyama, Caalim: *Invariant measure of rotational beta expansion and Tarski's Plank problem*. Discrete Comput. 681 Geom. 57, 357–370 (2017)
- Dombek, Masáková, Pelantová: *Number representation using generalized (-β)-transformation*. Theor. Comput. Sci. 412, 6653–6665 (2011)
- S.: *Representations for complex numbers with integer digits*. to appear in Res. Number Theory.

Thank you for your interest