Shift Radix Systems and Variations of Them

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Shift Radix System

Definition: Let $\mathbf{r} \in \mathbb{R}^d$ and

$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \dots, x_d) \to (x_2, \dots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor)$.

$\tau_{\mathbf{r}}$ is called shift radix system (SRS) if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

$$\begin{aligned} \mathcal{D}_d :=& \{\mathbf{r} \in \mathbb{R}^d | \forall \mathbf{x} \in \mathbb{Z}^d \exists n, l \in \mathbb{N} : \\ & \tau_{\mathbf{r}}^k(\mathbf{x}) = \tau_{\mathbf{r}}^{k+l}(\mathbf{x}) \ \forall k \geq n\} \\ \mathcal{D}_d^0 :=& \{\mathbf{r} \in \mathbb{R}^d | \tau_{\mathbf{r}} \text{ is SRS}\} \end{aligned}$$

Obviously $\mathcal{D}_d^0 \subset \mathcal{D}_d$.

Problem: Characterisation of $\mathcal{D}_d^0$ and $\mathcal{D}_d$.

Related Systems

β-**expansion:** (Rényi, Parry) Let $\beta \in \mathbb{R} \setminus \mathbb{Z}, \beta > 1$ and. Then $\gamma \in \mathbb{R}^+ \cup \{0\}$ has a unique representation of the form

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots$$

with

$$a_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\},\ 0 \le \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n.$$

Theorem: Let β be an algebraic number with minimal polynomial $(x - \beta)(x^{d-1} - r_{d-1}x^{d-2} - \cdots - r_2 x - r_1)$. The β-expansion is finite $\forall \gamma \in \mathbb{Z}[\frac{1}{\beta}] \cap [0, \infty) \Leftrightarrow (r_1, r_2, \ldots, r_{d-1}) \in \mathcal{D}^0_{d-1}$.

Canonical Number Systems: (Pethő) Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$ with $|p_0| \geq 2$ and $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ the quotient ring. Further let

$$x = X(P(X)Z[X]) \in R.$$

If every $A(x) \in R, A(x) \neq 0$ can be written in the form

$$A(x) = \sum_{i=0}^{n} a_i x^i, \ a_i \in \mathcal{N} := \{0, 1, \ldots, |p_0| - 1\},$$

then $(P(X), \mathcal{N})$ is called Canonical Number System (CNS) and P(X) an CNS Polynomial.

Theorem: $P(X)$ is an CNS Polynomial if and only if $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0$.

Properties of $\mathcal{D}_d$

Obviously $\mathcal{D}_1 = [-1, 1]$.

For $d \geq 2$:

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & -r_{d-1} & -r_d \end{pmatrix}$$

with $\mathbf{r} = (r_1, r_2, \ldots, r_d)$.

$$\mathcal{E}_d(\rho) := \{\mathbf{r} \in \mathbb{R}^d | \|R(\mathbf{r})\| < \rho\}$$

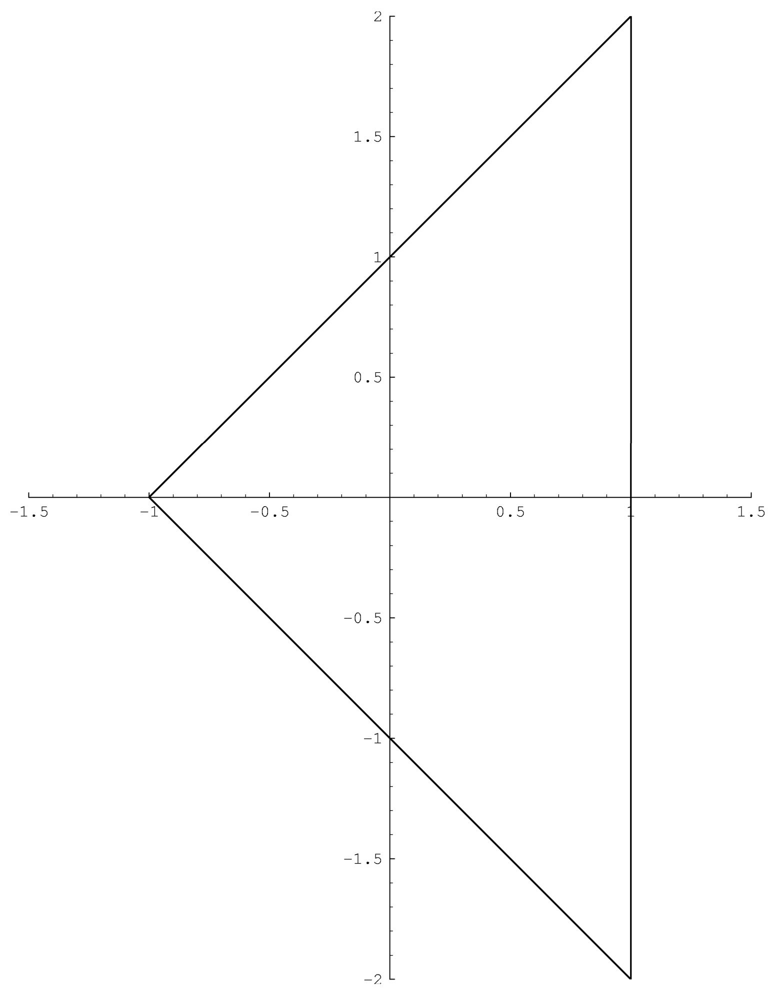
where $\| \cdot \|$ denotes the spectral norm.

Theorem: $\mathcal{E}_d(1) \subset D_d \subset \overline{\mathcal{E}_d(1)}$.

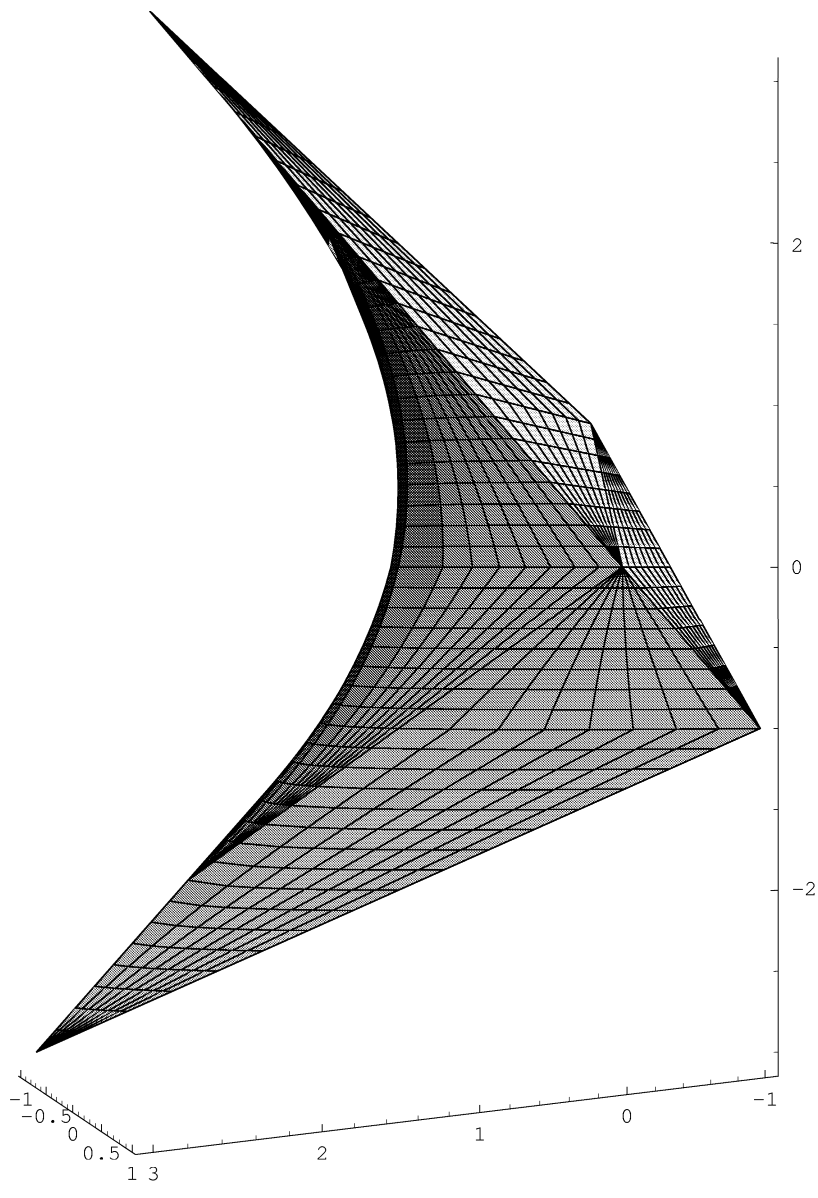
$$\mathcal{E}_2(1) = \{(x, y) \in \mathbb{R}^2 | |x| < 1, |y| < x + 1\}$$

$$\mathcal{E}_3(1) = \{(x, y, z) \in \mathbb{R}^3 | |x| < 1, \\ |y - xz| < 1 - x^2, |x + z| < |y + 1|\}$$

$\mathcal{D}_2$



$\mathcal{D}_3$



Constructing $\mathcal{D}_d^0$

- $\mathcal{D}_d^0 = \{(r_1, \ldots, r_d) \in \mathbb{R}^d | (0, r_1, \ldots, r_d) \in \mathcal{D}_{d+1}^0\}$

- We gain $\mathcal{D}_d^0$ by cutting out convex polyhedra from $\mathcal{D}_d$. Each polyhedron $P(\pi)$ corresponds to a period π of integers.

- **Theorem** (Brunotte): *For the convex hull $R \subset \mathcal{D}_d$ of points $\{r_1, \ldots, r_k\}$ with sufficiently small diameter there is an algorithm to find all the periods $\pi_j,\ j = 1, \ldots, k$, such that*

$$R \setminus \bigcup_{j=1}^{k} P(\pi_j) = \mathcal{D}_d^0 \cap R.$$

- It is possible to improve the algorithm such that convex R, which are bounded by curves, are allowed.

- Special methods are required for the analysis of areas near the boundary of $\mathcal{D}_d$.

Results for low dimensions

$d = 1$**:** It is easy to see that $\mathcal{D}_1^0 = [0, 1)$.

$d = 2$**: Example:** Let $\pi = -1, -1, 1, 2, 1$.

$$P(\pi) = \{\mathbf{r} \in \mathbb{R}^2 \mid \tau_{\mathbf{r}} : (-1,-1) \mapsto (-1,1) \mapsto (1,2) \mapsto (2,1) \mapsto (1,-1) \mapsto (-1,-1)\}$$

$P(\pi)$ is the solution of the system of inequalities

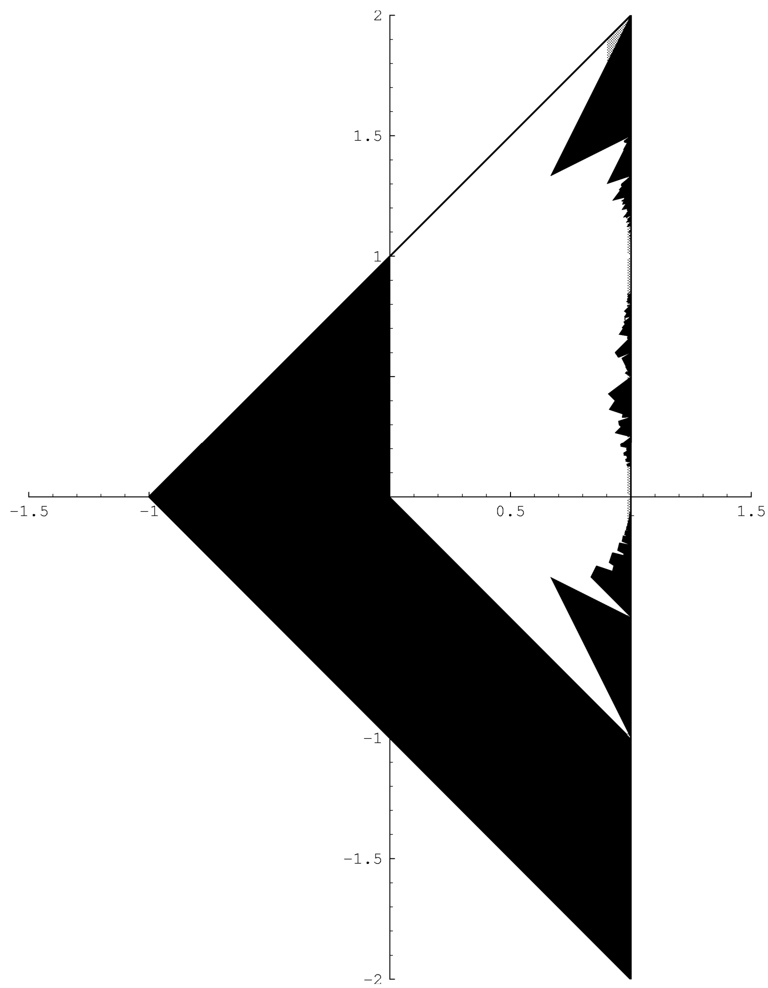
$$0 \leq a_i x + a_{i+1} y + a_{i+2} < 1, \ \ i = 1, \ldots, 5$$

with $a_1, \ldots, a_7 = -1, -1, 1, 2, 1, -1, -1$.

$$P(\pi) = \{(x, y) \in \mathbb{R}^2 \,| x \geq \frac{-y+1}{2}, x < -2y, x < y + 2\}$$

There are families of periods, which all yield nonempty cutout polyhedra. $\mathcal{D}_2^0$ cannot be constructed by finitely many cutouts.

$\mathcal{D}_2^0$



Symmetric Shift Radix Systems

Definition: Let $\mathbf{r} \in \mathbb{R}^d$ and

$$\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{r}\mathbf{x} + \frac{1}{2} \rfloor)$$

$\tilde{\tau}_{\mathbf{r}}$ is called symmetric shift radix system (SSRS) if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tilde{\tau}_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

Analogously we define

$$\begin{aligned} \tilde{\mathcal{D}}_d :=& \{\mathbf{r} \in \mathbb{R}^d | \forall \mathbf{x} \in \mathbb{Z}^d \exists n, l \in \mathbb{N} : \\ & \tilde{\tau}_{\mathbf{r}}^k(\mathbf{x}) = \tilde{\tau}_{\mathbf{r}}^{k+l}(\mathbf{x}) \ \forall k \geq n\} \\ \tilde{\mathcal{D}}_d^0 :=& \{\mathbf{r} \in \mathbb{R}^d | \tilde{\tau}_{\mathbf{r}} \text{ is SSRS}\} \end{aligned}$$

$$\mathcal{E}_d(1) \subset \tilde{\mathcal{D}}_d \subset \overline{\mathcal{E}_d(1)}$$

$$\tilde{\mathcal{D}}_1 = [-1, 1], \qquad \tilde{\mathcal{D}}_1^0 = (-\frac{1}{2}, \frac{1}{2}]$$

The methods for SRS to construct $\mathcal{D}_d^0$ can be transferred to SSRS.

$\tilde{\mathcal{D}}_2^0$

$\tilde{\mathcal{D}}_2^0$ is fully characterised (Akiyama, Scheicher):

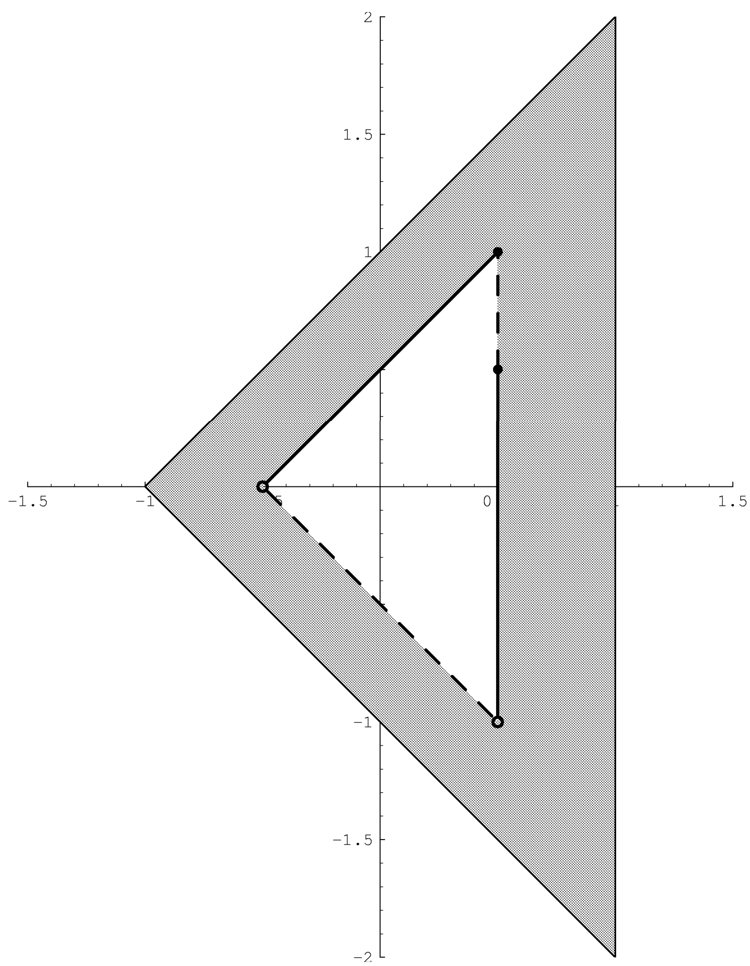
$$\tilde{\mathcal{D}}_2^0 = \overline{\frac{\mathcal{E}_2(1)}{2}} \setminus (L_1 \cup L_2)$$

with

$$L_1 = \{(x, y) \in \mathbb{R}^2 | |x| \leq \frac{1}{2}, y = -x - \frac{1}{2}\},$$

$$L_2 = \{(\frac{1}{2}, y) \in \mathbb{R}^2 | \frac{1}{2} < y < 1\}$$

$$\tilde{\mathcal{D}}_2^0$$



$\tilde{\mathcal{D}}_3^0$

$\tilde{\mathcal{D}}_3^0$ is partly characterised (together with Huszti, Scheicher, Thuswaldner):

$\tilde{\mathcal{D}}_3^0 \cap \partial \tilde{\mathcal{D}}_3 = \emptyset$. The set is away from the boundary. Finitely many cutout polyhedra suffice:

$$\tilde{\mathcal{D}}_3^0 = \tilde{\mathcal{D}}_3 \setminus \bigcup_{\pi \in \Pi} P(\pi)$$

with Π finite and for a sequence

$$x_1, \ldots, x_n \in \Pi \Rightarrow |x_i| \leq 2, i = 1, \ldots, n.$$

$\tilde{\mathcal{D}}_3^0$ consists of three connected convex bodies where some planes are attached.

$\tilde{\mathcal{D}}_3^0$ **(expected)**