New characterisation results for Shift Radix Systems

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Shift Radix System

Generalised definition: Fix $0 \le \epsilon \le \frac{1}{2}$. Let $\mathbf{r} \in \mathbb{R}^d$ and

$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{r}\mathbf{x} + \epsilon \rfloor)$.

$\tau_{\mathbf{r}}$ is called a shift radix system (SRS) if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

$\epsilon = 0$: classical SRS

$\epsilon = \frac{1}{2}$: symmetric shift radix systems (SSRS)

$$\begin{aligned} \mathcal{D}_d(\epsilon) := &\{\mathbf{r} \in \mathbb{R}^d | \forall \mathbf{x} \in \mathbb{Z}^d \exists n, l \in \mathbb{N} : \\ &\quad \tau_{\mathbf{r}}^k(\mathbf{x}) = \tau_{\mathbf{r}}^{k+l}(\mathbf{x}) \ \forall k \ge n\} \\ \mathcal{D}_d^0(\epsilon) := &\{\mathbf{r} \in \mathbb{R}^d | \tau_{\mathbf{r}} \text{ is SRS}\} \end{aligned}$$

Obviously $\mathcal{D}_d^0 \subset \mathcal{D}_d$.

Problem: Characterisation of $\mathcal{D}_d^0$ and $\mathcal{D}_d$.

Interpretation and Related Systems

β-**expansion:** (Rényi, Parry) Let $\beta \in \mathbb{R} \setminus \mathbb{Z}$, $\beta > 1$. Then any $\gamma \in [0, \infty)$ has a unique representation of the form

$$\gamma = a_m \beta^m + a_{m-1}\beta^{m-1} + \cdots$$

with

$$a_i \in \mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}, \ 0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n.$$

β has Property (F) $\Leftrightarrow$ β-expansion is finite for all $\gamma \in \mathbb{Z}[\frac{1}{\beta}]$ (only possible for Pisot-number (Frougny, Solomyak)).

Relation with SRS ($\epsilon = 0$):
Theorem 1. *Let β be a Pisot-number with minimal polynomial $(x - \beta)(x^{d-1} + r_{d-2}x^{d-2} + \cdots + r_2 x + r_0)$. Then β has property (F) if and only if $(r_0, r_2, \ldots, r_{d-2}) \in \mathcal{D}_d^0$.*

$\epsilon > 0$: this corresponds to a shift of the set of digits : $\mathcal{A} = \{\lfloor -\beta\epsilon \rfloor, \ldots, \lfloor \beta(1 - \epsilon) \rfloor\}$

Canonical Number Systems: (Pethő) Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$ with $|p_0| \geq 2$ and $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ the quotient ring. Further let

$$x = X(P(X)Z[X]) \in R.$$

If every $A(x) \in R$ can be written in the form

$$A(x) = \sum_{i=0}^{n} a_i x^i, \ a_i \in \mathcal{N} := \{0, 1, \ldots, |p_0| - 1\},$$

then $(P(X), \mathcal{N})$ is called a Canonical Number System (CNS) and P(X) a CNS Polynomial.

Relation with SRS ($\epsilon = 0$):

Theorem 2. *$P(X)$ is a CNS Polynomial if and only if $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0$.*

An $\epsilon > 0$ corresponds to a shift of the set of digits: $\mathcal{N} = \{n \in \mathbb{Z} | -\epsilon p_0 \leq n < (1 - \epsilon)p_0\}$

Visualization

$$\mathcal{E}_d(1) = \{(v_0, \ldots, v_{d-1}) \in \mathbb{R}^d \mid$$
$$\|x\| < 1 \forall x \text{ with } x^d + v_{d-1}x^{d-1} + \ldots + v_0 = 0\}$$

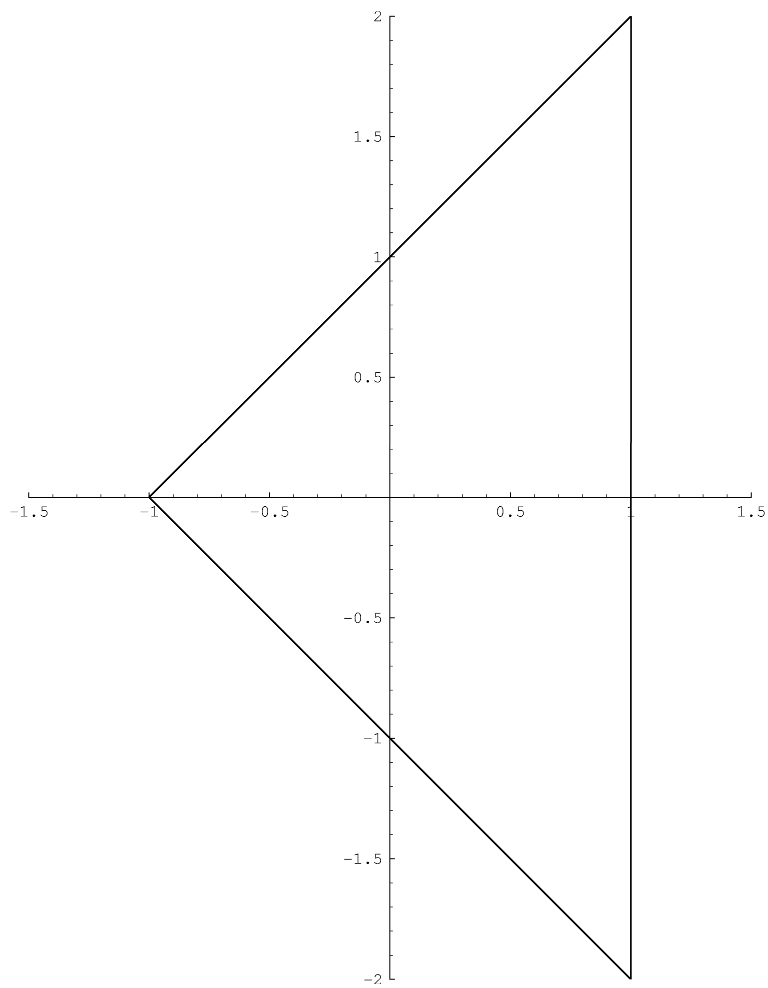
Theorem 3. $\mathcal{E}_d(1) \subset \mathcal{D}_d \subset \overline{\mathcal{E}}_d(1)$

$\mathrm{int}(\mathcal{D}_d(\epsilon_1)) = \mathrm{int}(\mathcal{D}_d(\epsilon_2))$ but in general $\partial\mathcal{D}_d(\epsilon_1) \neq \partial\mathcal{D}_d(\epsilon_2)$ for $\epsilon_1 \neq \epsilon_1$!
(Partial results concerning $\partial\mathcal{D}_2$ for $\epsilon = 0$ from Akiyama, Brunotte, Pethő, Steiner)

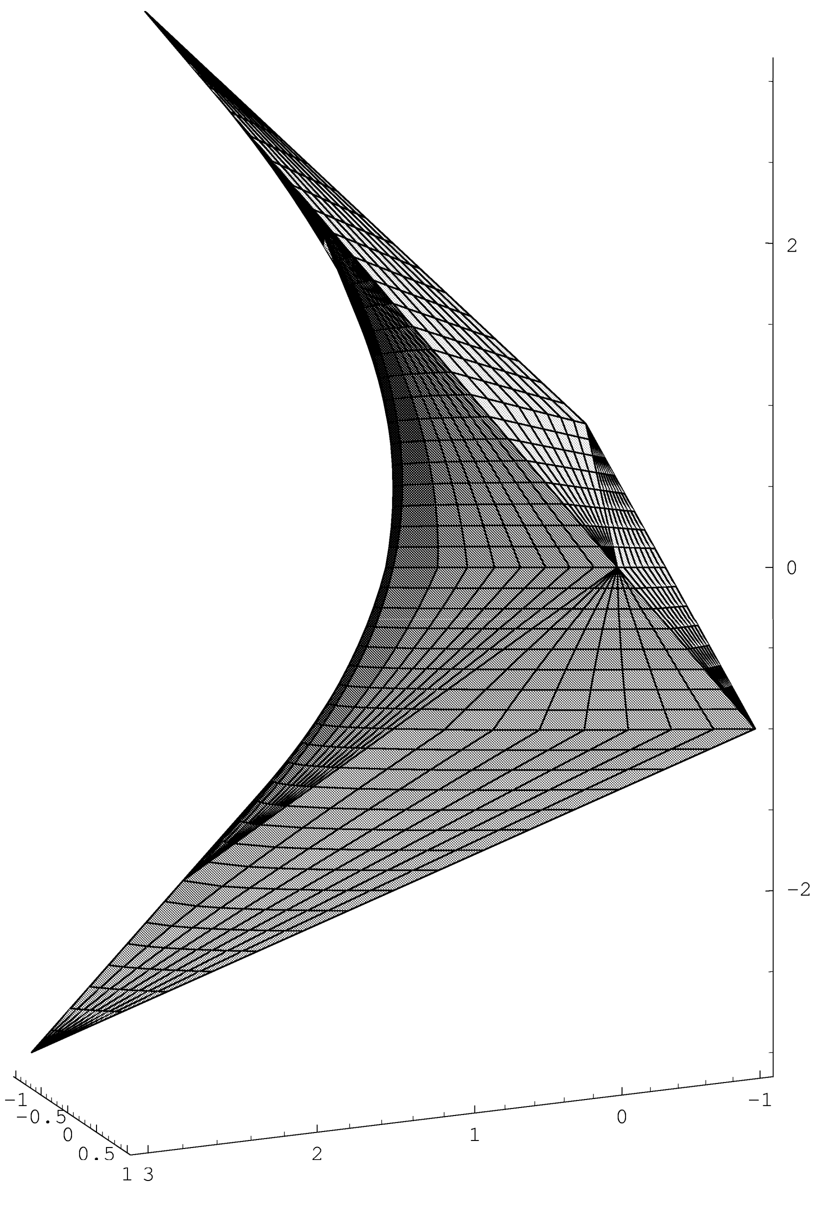
The sets $\mathcal{E}_d(1)$ are known (Schur, Takagi)

$$\mathcal{E}_2(1) = \{(x, y) \in \mathbb{R}^2 || x| < 1, |y| < x + 1\}$$
$$\mathcal{E}_3(1) = \{(x, y, z) \in \mathbb{R}^3 || z| < 3,$$
$$y - xz < 1 - x^2, |x + z| < y + 1\}$$

$\mathcal{D}_2$



$\mathcal{D}_3$



$\mathcal{D}_d^0$ is gained by cutting out polyhedra from $\mathcal{D}_d$. A polyhedron corresponds to a period of $\tau_{\mathbf{r}}$ of integers.

A period $a_0, \ldots, a_{l-1}$ of length l induces a system of inequalities

$$0 \leq a_i x_1 + \ldots, a_{i+d-1} x_d + \epsilon < 1, i = 0, \ldots, l-1$$

with the indices of a taken modulo l.
Such a system describes a (possibly degenerated) polyhedron.

$$\mathcal{D}_d^0 = \{(r_1, \ldots, r_d) \in \mathbb{R}^d | (0, r_1, \ldots, r_d) \in \mathcal{D}_{d+1}^0\}$$

Current situation

$\mathcal{D}_2^0, \epsilon = 0$**:** (results of Akiyama, Brunotte, Pethő, Thuswaldner and Surer)

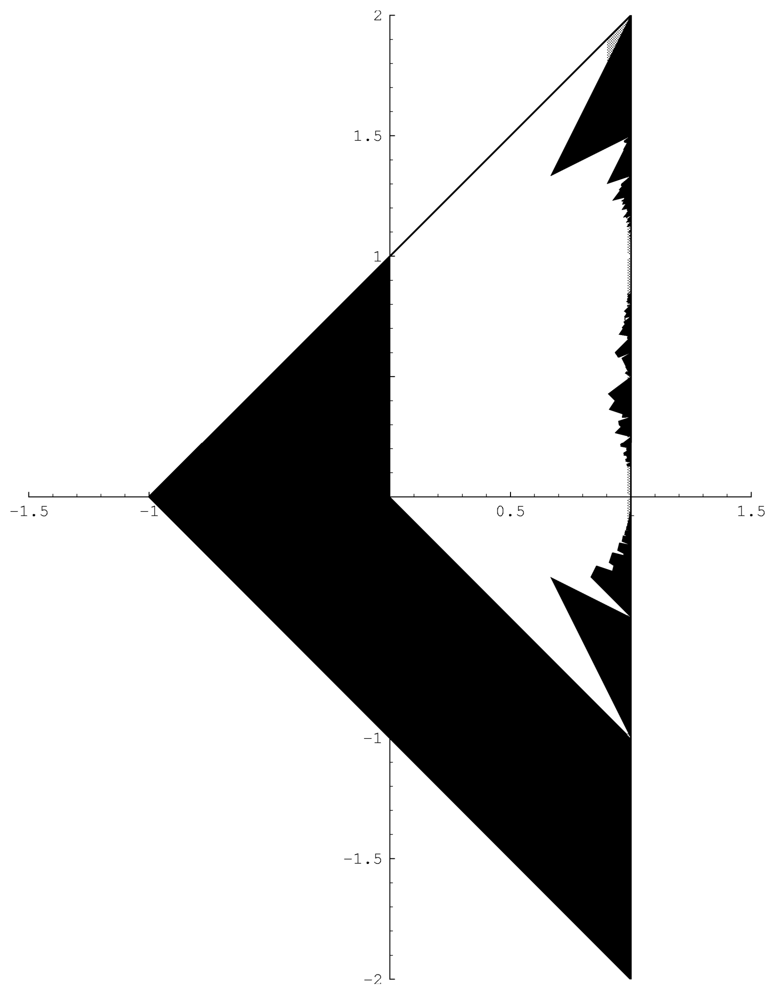
- We know more than 500 cutout polyhedra fully characterising more than 98% of the entire area. Problem: regions near the right boundary.

- The points $(1, 0)$ and $(1, 1)$ are **critical points** (points, where any neighbourhood cannot be described by cutting only finitely many polyhedra).

- There are two explicitly known infinite families of cutouts.

- There are periods of arbitrary length and arbitrary size of their entries.

$$\mathcal{D}_2^0, \epsilon = 0$$



Current situation

$\mathcal{D}_2^0, \epsilon = \frac{1}{2}$**:**
fully characterised (Akiyama, Scheicher)

$$\frac{\overline{\mathcal{E}_2(1)}}{2} \setminus (L_1 \cup L_2)$$

with

$$L_1 = \{(x,y) \in \mathbb{R}^2 | |x| \leq \frac{1}{2}, y = -x - \frac{1}{2}\},$$

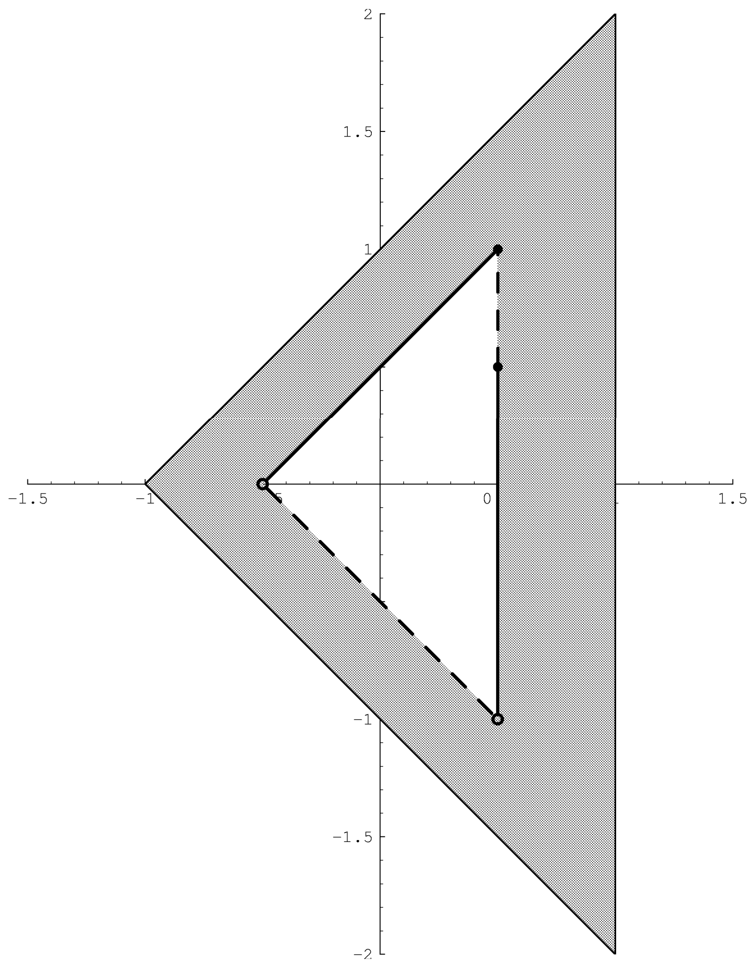
$$L_2 = \{(\frac{1}{2}, y) \in \mathbb{R}^2 | \frac{1}{2} < y < 1\}$$

- full characterisation possible because the set $\mathcal{D}_2^0$ is away from the boundary of $\mathcal{D}_2$.

- 9 cutout polyhedra are sufficient

- The lengths l of the corresponding periods do not exceed 12, the maximum of the absolutes of the entries is 2 at most.

$$\mathcal{D}_2^0, \epsilon = \frac{1}{2}$$

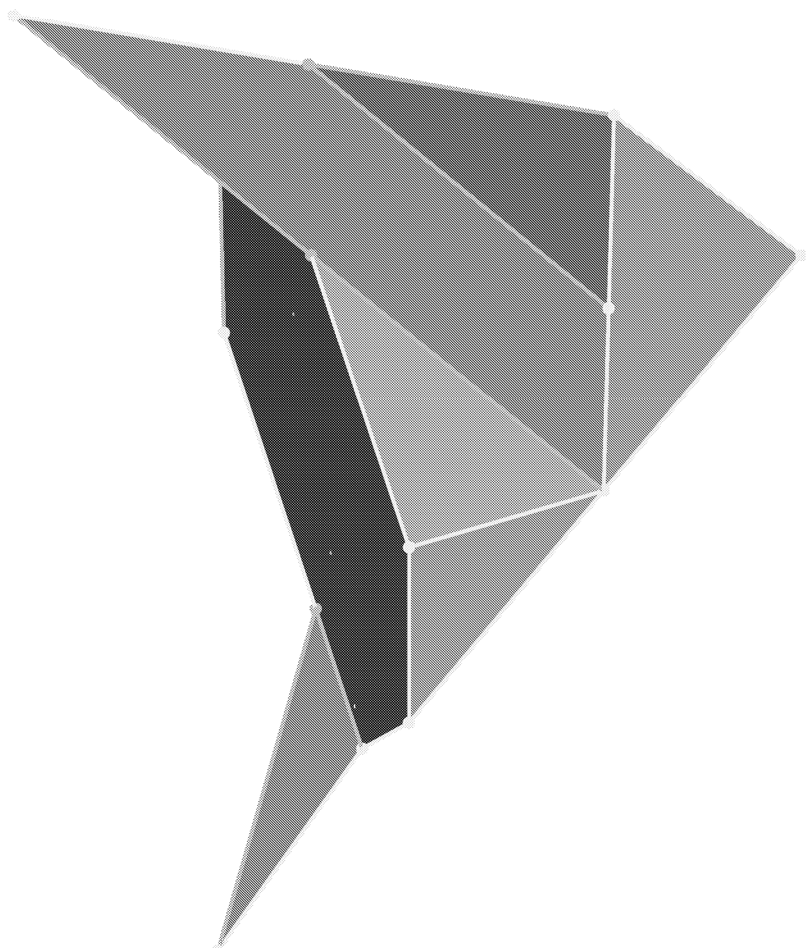


Current situation

$\mathcal{D}_3^0, \epsilon = \frac{1}{2}$**:**
fully characterised (Huszti, Scheicher, Surer, Thuswaldner)

- full characterisation possible because the set $\mathcal{D}_3^0$ is away from the boundary of $\mathcal{D}_3$.
- composition of three convex bodies
- 43 cutout polyhedra are sufficient
- The lengths l of the corresponding periods do not exceed 22, the maximum of the absolutes of the entries is 2 at most.

$$\mathcal{D}_3^0, \epsilon = \frac{1}{2}$$



Some open questions

- Classification of the critical points ($\epsilon = 0$).

- Relationship between the length l and $\sum_{i=0}^{l-1} a_i$ of a period.

- Are finitely many cutouts sufficient for $n > 3$ ($\epsilon = \frac{1}{2}$)?

- Situation for $\epsilon \in (0, \frac{1}{2})$. Is the mapping $\epsilon \mapsto \mathcal{D}_d^0(\epsilon)$ continuous in ϵ?.