CODING OF SUBSTITUTION DYNAMICAL SYSTEMS

PAUL SURER

Short abstract

It is well known that the elements of a substitution dynamical system Ω can be coded as combinatorial power series. The "digits" are the labels of a finite graph (prefix-suffix graph) and the digit sequences are given by the infinite walks. The aim of the talk is to present a much more general setting. In fact, it is possible to code Ω in many analogue ways by using different sets of combinatorial digits. The known prefix-suffix representation then is only one special case but we will see that many of the results can be generalised without problems. I also want to present applications of the theory and planned sequel research.

Extended abstract

1. Motivation

At first I want to present why we need an alternative coding for substitution dynamical system. This shall be done by the following motivative example. Denote by β the real root of $x^3 - 2x^2 - 1$. β is a Pisot unit. It is well-known (see, for example, [4]) that the induced beta-shift is a shift of finite type and closely related to the substitution dynamical system induced by the beta substitution $\sigma_\beta : 1 \mapsto 112, 1 \mapsto 3, 3 \mapsto 1$. This can be seen quite easily by comparing the presentation graph of the beta-shift and the prefix-suffix graph of the substitution. Geometrically the relation can be visualised by comparing the central tile induced by the beta-expansion (in the spirit of [1]) and the Rauzy Fractal induced by σ_β. The tiles coincide (possibly up to a linear factor depending on the exact way of construction). Figure 1 shows the two tiles for our example.

Figure 1. The central tile induced by the β-expansion (left) and the Rauzy fractal induced by substitutions σ_β (right) with its natural subdivision into three subtiles.

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Akiyama and Scheicher [2] introduced symmetric beta-expansions and gave necessary and sufficient conditions for the induced symmetric beta-shift to be sofic. For β chosen as above the symmetric beta-shift is of finite type since the symmetric beta-expansion of $-\frac{1}{2}$ is purely periodic (the digit sequence equals $(-1,0,0)^\infty$). Analogously to the classic beta-expansion one can define a central tile for the symmetric beta-expansion (see [8]). Figure 2 (left) shows this tile for our example. Right beside we see the Rauzy fractal associated to the substitution $\sigma'_\beta : 1 \mapsto 121, 1 \mapsto 3, 3 \mapsto 1$. The coincidence between the two fractal tiles is amazing. Since similar settings match analogously we can exclude an accidental similarity. Thus, there must be an up to now unknown relationship between the two types of dynamical systems. The minimal right resolving presentation graph (in the sense of [9])

FIGURE 2. The central tile induced by the symmetric β–expansion (left) and the Rauzy fractal induced by substitutions σ'_β (right) with its natural subdivision into three subtiles.

of the symmetric beta shift for our example has 5 vertices while the prefix-suffix graph of σ'_β has 3 vertices only. Hence, the familiar way of coding the substitution dynamical system does not seem to explain this correspondence.

2. Notation

For the rest of this abstract let σ be a primitive, not shift-periodic substitutions over the alphabet $\mathcal{A}$, Ω the induced (two-sided) substitution dynamical systems and S the left-shift. We denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$ where ε denotes the empty word. For an $A \in \mathcal{A}^*$ define $|A|$ to be the length of the word A. The set $\mathcal{L}_n$ for $n \geq 1$ denotes the set of words of length n that appear in the elements of Ω.

Let $w \in \Omega$. Then $w_j \in \mathcal{A}$ denotes the letter of w indexed by $j \in \mathbb{Z}$. For $i \leq j$ set $w_{[i,j]} = w_i w_{i+1} \cdots w_j \in \mathcal{A}^*$. If $j < i$ we set $w_{[i,j]} = \varepsilon$. We write $w_{(-\infty,j]}$ for the left-infinite word $\cdots w_{j-2} w_{j-1} w_j$ and, analogously, $w_{[j,\infty)}$ for the right-infinite word $w_j w_{j+1} w_{j+2} \cdots$. Finally, we use the symbol . to mark the position of the last negative index (and the first non-negative index, respectively). Thus, $w = w_{(-\infty,-1]}.w_{[0,\infty)}$.

3. Main results

Due to [6, 7] the elements of Ω can be coded as infinite paths of a finite graph (prefix-suffix graph). The main idea I want to present in the talk is a more general way of coding substitution dynamical systems as shift of finite type. Previous results are based on the following theorem.

Theorem 3.1 (*cf.* [10])**.** *Let* $w \in \Omega$. *Then there is a unique pair* $(u,k) \in \Omega \times \mathbb{N}$ *with* $0 \le k < |\sigma(u_0)|$ *such that*

$$w = S^k \sigma(u).$$

We will generalise this theorem by also allowing right-shifts S^{-1} for the coding. Since we use bi-infinite sequences (in fact, Theorem 3.1 requires the use of bi-infinite sequences), S^{-1} is well defined. To maintain the coding unique we give a prescription when we use the left-shift and when we the right-shift.

Definition 3.2. We call a finite collection

$$\mathcal{N}_\sigma = \Big\{ \mathcal{N}(a) \subset \{-|\sigma(a)|+1, \dots, |\sigma(a)|-1\} \Big| \\ a \in \mathcal{A}, |\mathcal{N}(a)| = |\sigma(a)|, \forall (k,n) \in \mathcal{N}(a)^2 : k \not\equiv n \,(\mathrm{mod}\ |\sigma(a)|) \Big\}$$

a *coding prescription* of (Ω, S).

$\mathcal{N}_\sigma$ consists of $|\mathcal{A}|$ finite sets $\mathcal{N}(a)$, one for every letter $a \in \mathcal{A}$. A set $\mathcal{N}(a)$ consists of a maximal set of representatives of the integers modulo $|\sigma(a)|$ whose modulus is bounded by $|\sigma(a)|-1$. Consequently $\mathcal{N}(a)$ includes 0 for every $a \in \mathcal{A}$. But there is still a lot of freedom left, depending on the substitution. One easily calculates that for a given substitution σ there are exactly

$$\prod_{a \in \mathcal{A}} 2^{|\sigma(a)|-1} > 1$$

different coding prescriptions. For each set $\mathcal{N}(a) \in \mathcal{N}_\sigma$ denote by $\mathcal{N}(a)^+$ ($\mathcal{N}(a)^-$, respectively) the subset of non-negative (non-positive, respectively) elements of $\mathcal{N}(a)$[1]. Hence, $\mathcal{N}(a)^+ \cup \mathcal{N}(a)^- = \mathcal{N}(a)$, $\mathcal{N}(a)^+ \cap \mathcal{N}(a)^- = \{0\}$ for every $a \in \mathcal{A}$. Now, the following theorem holds.

Theorem 3.3. *Let* $w \in \Omega$ *and* $\mathcal{N}_\sigma$ *a coding prescription of* (Ω, S). *Then there is a unique pair* $(\Theta(w), \varphi(w)) \in \Omega \times \mathbb{Z}$ *such that* $\varphi(w) \in \mathcal{N}(\Theta(w)_{-1})^- \cup \mathcal{N}(\Theta(w)_0)^+$ *and*

$$w = S^{\varphi(w)} \sigma(\Theta(w)).$$

In particular, the theorem shows that there are a lot of ways to desubstitute the elements of Ω uniquely.

Analogously to [6, 7], by successive application of Theorem 3.3, we can represent every element of Ω as an infinite digit sequence. Of course, the digit set is intimately related to the decoding prescription. Thus, we have a lot of freedoms. Actually, there are several (equivalent) ways of defining the digits. Here we use inverse letters. Such inverses appear in a much more general setting, for example, in [3]. For every element a of $\mathcal{A}$ we denote by $\overline{a}$ the inverse of a by concatenation. Let $\tilde{\mathcal{A}}$ be the extended alphabet (consisting of the letters of $\mathcal{A}$ and its inverses). The set of finite words over $\tilde{\mathcal{A}}$ is denoted by $\tilde{\mathcal{A}}^*$ and form

[1]Here we differ from the convention that a supscript "+" ("−") denotes the strictly positive (negative) elements of a set

with the concatenation a free group (with neutral element ε). The most important rules are:

- $\overline{AB} = \overline{B}\,\overline{A}$ for all $A, B \in \tilde{A}^*$;
- $\overline{(\overline{A})} = A$ for all $A \in \tilde{A}^*$;
- $A\overline{A} = \varepsilon$ for all $A \in \tilde{A}^*$;
- $\overline{\varepsilon} = \varepsilon$.

We can easily extend σ to $\tilde{\mathcal{A}}$ by defining $\sigma(\overline{a}) = \overline{\sigma(a)}$. Hence, we have $\sigma\overline{A} = \overline{\sigma A}$ for all $A \in \tilde{A}^*$. With this notation we define the set

$$\mathcal{R} := \{(a.b, d) \mid \exists k \in \mathcal{N}(a)^- \cup \mathcal{N}(b)^+, \sigma(ab)_{[1,|\sigma(a)|+k]} = \sigma(a)d\}.$$

Note that for some $(a.b, d) \in \mathcal{R}$ the word $d \in \mathcal{A}^*$ consists either only of letters of $\mathcal{A}$ (if the respective k is positive) or only of inverse letters (if $k < 0$) or is the empty word ε (if $k = 0$). Define

$$\Gamma : \Omega \longrightarrow \mathcal{R}^\infty, w \mapsto (\Theta^n(w)_{-1}.\Theta^n(w)_0, d_n)_{n\geq 1}$$

where, for every $n \geq 1$, d_n is the (uniquely determined) element of $\tilde{\mathcal{A}}^*$ with $\Theta^{n-1}(w) = \sigma(\Theta^n(w)_{(-\infty,-1]})d_n.\overline{d_n}\sigma(\Theta^n(w)_{[0,\infty)})$. With the convention that inverse letters give negative contributions to the length of a word we obviously have $|d_n| = \varphi(\Theta^{n-1}(w))$.

Similar to other papers we can associate the process of coding to a finite graph. But now we cannot use the alphabet $\mathcal{A}$ (hence, $\mathcal{L}_1$) as set of vertices. We will use $\mathcal{L}_2$ as set of vertices.

Definition 3.4. Let $\mathcal{N}_\sigma$ a decoding prescription of (Ω, S). The *coding graph associated to σ with respect to $\mathcal{N}_\sigma$*, denoted by $G(\sigma, \mathcal{N}_\sigma)$, is the graph with the following properties: The vertices of $G(\sigma, \mathcal{N}_\sigma)$ are pairs $(a.b) \in \mathcal{A} \times \mathcal{A}$ such that $ab \in \mathcal{L}_2$. There is an edge from $(a.b)$ to $(a'.b')$ labelled by $(a'.b', d) \in \mathcal{R}$ if $(\sigma(a')d)_{|\sigma(a')d|} = a$ and $(\overline{d}\sigma(b'))_1 = b$.

Then we have the following theorem.

Theorem 3.5. *The coding map $\Gamma_{\mathcal{R}}$ maps Ω surjectively onto the infinite walks of $G(\sigma, \mathcal{N}_\sigma)$. The map is one-to-one with the the S-orbits or the S^{-1}-orbits of the periodic points of σ as possible exceptions. Here the map is finite-to-one.*

We see that the prefix-suffix representation corresponds exactly to the case when the coding prescription consists of non-negative integers only. On the base of examples the theory shall be explained. I want to show differences and advantages to previous results. Most of the statements of [6, 7] can be restated in this generalised framework (periodicity of the periodic points, possibility of decoding the original $w \in \Omega$, adic transformation on the infinite walks, etc.). We will also see that several properties of the coding depend on the coding prescriptions.

4. Applications

Based on examples I want to present several applications of the theory and possibilities of sequel researches.

Numeration: By choosing the correct coding prescription we are now able to understand the relation between symmetric beta-expansions and substitutions. It is to expect that analogue results hold for epsilon-beta-expansion (in the sense of [11]) and, more generally, that for a given Pisot substitution each coding prescription induces a numeration system as a generalisation of the Dumont-Thomas numeration.

Construction of Rauzy fractals: We can use the coding graph associated to a unit Pisot substitution for constructing the Rauzy fractal. Each coding prescription gives a different way of construction but the Rauzy fractal does not change (since the underlying dynamical system is always the same). Since the graph has more vertices we have a refined subdivision into subtiles. This refinement explains the action of the shift map on the Rauzy fractal and its subtiles.

Induced substitutions: A coding graph for a given substitution σ induces (at least for some coding prescriptions) another substitution over the alphabet $\mathcal{L}_2$ which has the same Perron-Frobenius eigenvalue as σ (hence, such substitutions are never irreducible). Different coding prescriptions give different substitutions. Comparison of these induced substitutions with (for the same original substitution σ) show interesting effects.

Automorphisms of the free group: We already used inverse letters for the representation. Thus, it is straightforward to apply the theory on dynamical systems induced by iwip-automorphisms of the free group (see, for example, [5, 3]).

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Departamento de Matemática, IBILCE - UNESP, Rua Cristóvão Colombo, 2265 - Jardim Nazareth, 15.054-000 São José do Rio Preto - SP, BRAZIL
E-mail address: me@palovsky.com