DYNAMICAL PROPERTIES OF THE TENT MAP

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Let $\beta > 1$ and define the tent map $T_\beta : [0,1] \longrightarrow [0,1]$ by

$$T_\beta(x) = \begin{cases} \beta x & \text{for } x \in [0, \frac{1}{\beta}], \\ \frac{\beta}{\beta-1}(1-x) & \text{for } x \in [\frac{1}{\beta}, 1]. \end{cases}$$

The tent map T_β induces a symbolic dynamical system, which is the full binary shift, and allows us to represent each $x \in [0,1]$ as

$$x = \sum_{0 \leq k < M} (-\beta')^{-k} \beta^{-(l_0 + \cdots + l_k)},$$

where $\beta' = \beta(\beta-1)^{-1}$, $M \in \mathbb{N} \cup \{\infty\}$, and $l_0, l_1, \ldots$ is a uniquely determined (finite or infinite) sequence of non-negative integers (*cf.* [2]).

Analogously to other types of expansions we may ask for bases β such that the set of numbers with periodic or even finite representation is as large as possible. In particular, we say that T_β satisfies the periodicity property (P) if the orbit $(T_\beta^n(x))_{n\geq 1}$ is eventually periodic for all $x \in \mathbb{Q}(\beta) \cap [0,1]$, and T_β has the finiteness property (F) whenever the T_β-orbit of each element of $\mathbb{Q}(\beta) \cap [0,1]$ contains 0.

The principle intention of the talk is to present result concerning (P) and (F). At the beginning we summarise several well-known facts published by Lagarias *et al.* in [2, 3]. The focus is on a recently discovered connection between tent maps and generalised beta-transformations that allows us to transfer results concerning periodicity and finiteness properties (see [4]).

Define the set

$$\mathfrak{G} := \left\{ \beta \in (1,2] \;\middle|\; \exists P \in \mathbb{N} : \left(\frac{\beta}{\beta-1}\right)^2 = \beta^P \right\}.$$

For $\beta > 1$ let $\xi := \beta(2\beta-1)^{-1}$ be the (non-zero) fixed point of T_β. Define the interval $I := [\xi - 1, \xi)$ and the modified fractional part function

$$\phi : \mathbb{R} \longrightarrow I, \;\; x \longmapsto x - \lfloor x + (1-\xi) \rfloor.$$

Observe that the restriction of ϕ on the unit interval $[0,1]$ is almost bijective, *i.e.* bijective up to a set of measure zero. Denote by τ_β the modified beta-transformation with respect to the base β

$$\tau_\beta : I \longrightarrow I, \;\; x \longmapsto \phi(\beta x).$$

With these notations we can state the following relation, which is a weaker form of measure-theoretical conjugacy of dynamical systems.

Theorem (*cf.* [4])**.** *Let $\alpha \in \mathfrak{S}$. Then for each $x \in [0,1]$ there exist positive integers $p = p(x), q = q(x)$ such that $\phi \circ T_\beta^p(x) = \tau_\beta^q \circ \phi(x)$.*

Observe that there exists no upper bound for p and q, *i.e.* for arbitrary large $N \in \mathbb{N}$ we can find an open interval $J \subset [0,1]$ such that for each $x \in J$ we have $\phi \circ T_\alpha^p(x) \neq \tau_\alpha^q \circ \phi(x)$ for all $1 \leq p, q \leq N$.

With this theorem it is possible to use known results concerning modified beta-transformations (see [1, 5]) in order to obtain informations about tent maps that satify (F). However, the relation turns out to be too weak for the underlying subshifts to be connected via a finite state transducer.

At the end of the talk we want to outline further research projects in context with tent maps as, for example, geometric (fractal) representations.

References

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