# SYMMETRIC AND CONGRUENT RAUZY FRACTALS

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*Based on a joint work with K. Scheicher and V. Sirvent*

Let $\mathcal{A} = \{1, 2, \dots, m\}$ be a finite set (alphabet) and $\mathcal{A}^*$ the finite words over $\mathcal{A}$. We denote by $\sigma$ a morphism $\mathcal{A}^* \longrightarrow \mathcal{A}^*$ that we require to be primitive, hence, there exists a power $n$ such that for each two letters $i, j \in \mathcal{A}$ the word $\sigma^n(j)$ contains $i$ at least once.

Primitivity ensures that the matrix $M_\sigma = (|\sigma(j)|_i)_{1\leq,i,j\leq m}$ (where $|\sigma(j)|_i$ denotes the number of occurrences of $i$ in the word $\sigma(j)$) possesses a dominant real eigenvalue $\beta$. We are interested in unimodular Pisot substitutions, that is $\beta$ is a Pisot unit of algebraic degree $d+1$ (thus, $d+1 \leq m$). In this case it is well known that we can associate to $\sigma$ a compact set $\mathcal{R}_\sigma \subset \mathbb{R}^d$ of fractal shape known as Rauzy fractal (see, for example, [2, 3]).

Given unimodular Pisot substitutions $\sigma$, $\tau$ we are interested in the following problems:

- Which conditions ensure the associated Rauzy fractals $\mathcal{R}_\sigma$, $\mathcal{R}_\tau$ to be congruent (that is they differ by an affine transformation only)?
- Which conditions ensure that the Rauzy fractal $\mathcal{R}_\sigma$ is central symmetric?

We discuss the questions in terms of the induced language. For a primitive substitution $\sigma$ it is defined by

$$\mathcal{L}_\sigma := \{A \in \mathcal{A}^* |\, \exists n \geq 1 : A \text{ is a subword of } \sigma^n(1)\}.$$

Therefore, our discourse involves a lot of combinatorics on words. Concretely we have the following two main results.

**Theorem** (*cf.* [5])**.** *Let $\sigma$ and $\tau$ be irreducible Pisot substitutions (i.e. $d+1 = m$) over the same alphabet $\mathcal{A}$. If $\mathcal{L}_\sigma = \mathcal{L}_\tau$ then the Rauzy fractals $\mathcal{R}_\sigma$, $\mathcal{R}_\tau$ are congruent.*

For irreducible substitutions we need additional conditions.

For a word $A \in \mathcal{A}^*$ we denote by $\tilde{A}$ the reversed word (or *mirror-word*). We call a set $\mathcal{L} \subset \mathcal{A}^*$ *mirror-invariant* if for each $A \in \mathcal{L}$ we have $\tilde{A} \in \mathcal{L}$. With these notations we can state the following theorem concerning central symmetric Rauzy fractals.

**Theorem** (*cf.* [5])**.** *Let $\sigma$ be a Pisot substitutions over $\mathcal{A}$. If $\mathcal{L}_\sigma$ is mirror-invariant then the Rauzy fractals $\mathcal{R}_\sigma$ is central-symmetric.*

Based on these two theorems we will present concrete classes of substitutions that yield congruent or central-symmetric Rauzy fractals. For these classes we will be able to give explicit expressions for the respective translation and the point

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of symmetry, respectively. Furthermore, we analyse whether the conditions in our theorems are necessary. We will see that the topic is related with up to now unsolved problems as the *Pisot conjecture* (see for example [1]) and the *Class $\mathcal{P}$ conjecture* stated in [4]. The presentation will be accompanied by illustrative examples.

## References


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