Substitutions, Shifts of Finite Type, and Numeration*

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Let $\mathcal{A}$ be a finite set (alphabet), $\mathcal{A}^*$ the free monoid over $\mathcal{A}$, and $\mathcal{A}^{\mathbb{Z}}$ the set of bi-infinite words over $\mathcal{A}$. For a sequence $w = (w_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ we especially tag the *central pair* $w_{-1}w_0$ which will be denoted by a point, *i.e.* $w = \cdots w_{-3}w_{-2}w_{-1}.w_0w_1w_2\cdots$.

Denote by $\sigma : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ a non-erasing morphism (substitution). Suppose that σ is primitive, that is there exists an $n \in \mathbb{N}$ such that for all $a, b \in \mathcal{A}$ the letter b appears in $\sigma^n(a)$ at least once. Extend σ to $\mathcal{A}^{\mathbb{Z}}$ by defining

$$\sigma(\cdots w_{-3}w_{-2}w_{-1}.w_0w_1w_2\cdots) = \cdots\sigma(w_{-3})\sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\sigma(w_2)\cdots.$$

Denote by $\mathfrak{L}$ the induced language given by

$$\mathfrak{L} := \{A \in \mathcal{A}^* |\, \exists a \in \mathcal{A}, n \in \mathbb{N} \text{ such that } A \text{ is a subword of } \sigma^n(a)\}.$$

The *substitution dynamical system induced by* σ is the pair (Ω, S), where

$$\Omega = \left\{(w_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} |\, \forall i \in \mathbb{Z}, k \in \mathbb{N} : w_i \cdots w_{i+k} \in \mathfrak{L}\right\}$$

and S is the left shift

$$S : \cdots w_{-3}w_{-2}w_{-1}.w_0w_1w_2\cdots \mapsto \cdots w_{-3}w_{-2}w_{-1}w_0.w_1w_2\cdots.$$

The principle aim of the talk is to present how to code substitution dynamical systems as shifts of finite type with respect to a so-called *coding prescription*. According to [4], a coding prescription is a function c with source $\mathcal{A}$ that assigns to each letter $a \in \mathcal{A}$ a complete set of representatives modulo $|\sigma(a)|$ whose absolute values are smaller than $|\sigma(a)|$ (where $|\sigma(a)|$ denotes the length of the word $\sigma(a)$). We extend c to $\mathcal{A}^2$ by

$$c(ab) := \{k \in c(a) |\, k \leq 0\} \cup \{k \in c(b) |\, k \geq 0\}.$$

A coding prescription induces in a natural way a finite graph $G(c)$. The vertices are given by $\mathfrak{L}_2 := \mathfrak{L} \cap \mathcal{A}^2$. There is an edge from $a'b'$ to ab labelled by (ab, k) if $k \in c(ab)$ and $a'b'$ appears at the $(k + |\sigma(a)|)$th position in $\sigma(ab)$. We will see that Ω can be coded continuously and surjectively as the shift of finite type given by the infinite walks on $G(c)$. Depending on the actual coding prescription, several interesting effects may occur (for example in context with the injectivity). The special case where $c(a)$ does not contain negative integers for all $a \in \mathcal{A}$ corresponds to the well-known prefix-suffix coding (see [1, 3]).

It is well known that substitutions are intimately related with numeration (Dumont-Thomas numeration [2]). The concept of coding prescriptions allows us to generalise these results. We will see that different coding prescriptions correspond to different sets of digits.

*Supported by the FWF.

References

[1] V. Canterini and A. Siegel, *Automate des préfixes-suffixes associé à une substitution primitive*, J. Théor. Nombres Bordeaux, 13 (2001), pp. 353–369.

[2] J.-M. Dumont and A. Thomas, *Systemes de numeration et fonctions fractales relatifs aux substitutions*, Theoret. Comput. Sci., 65 (1989), pp. 153–169.

[3] C. Holton and L. Q. Zamboni, *Directed graphs and substitutions*, Theory Comput. Syst., 34 (2001), pp. 545–564.

[4] P. Surer, *Coding of substitution dynamical systems as shifts of finite type*, Ergodic Theory Dynam. Systems, 36 (2016), pp. 944–972.