Non-monic digit systems, Part II

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Definitions

- $\mathbb{E}$ a algebraic extension of $\mathbb{Q}$ of degree g,
- $\mathcal{E}$ the ring of integers of $\mathbb{E}$,
- $P = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_1 x + p_0 \in \mathcal{E}[x]$,
- $\mathcal{R} = \mathcal{E}[x]/P(x)\mathcal{E}[x]$,
- $\pi : \mathcal{E}[x] \to \mathcal{R}$ the canonical epimorphism,
- $X := \pi(x)$ the image of X under the canonical epimorphism,
- $\mathcal{N}$ a set of representatives of $\mathcal{E}/p_0$ (digit set), $0 \in \mathcal{N}$,
- $|\mathcal{N}| = N(p_0)$ where $N(p_0)$ is the algebraic norm of p_0.

Definitions

Definition

An $A \in \mathcal{R}$ has a finite X-ary representation if there are $e_0, \ldots, e_h \in \mathcal{N}$ such that

$$A = \sum_{i=0}^{h} e_i X^i$$

and $e_h \neq 0$. The pair $(P, \mathcal{N})$ is called *digit system* in $\mathcal{R}$ if each $A \in \mathcal{R}$ has a finite X-ary representation.

The backward division algorithm

The mapping T

$$T : \mathcal{E}[x] \to \mathcal{E}[x],$$
$$\sum_{i=0}^{n} a_i x^i \to \sum_{i=0}^{n-1} a_{i+1} x^i - q \sum_{i=0}^{d-1} p_{i+1} x^i$$

where $q = \frac{a_0 - e}{p_0} \in \mathcal{E}$ with (uniquely determined) $e \in \mathcal{N}$.

Theorem

Let $Q \in \mathcal{E}[x]$. $\pi(Q)$ has a finite X-ary expansion if and only if there exists a $k \in \mathbb{N}$ such that $T^k(Q) = 0$.

Special case: CNS ($g = 1$)

We have

- $\mathcal{E} = \mathbb{Z}$,
- $\mathcal{N}(p_0) = \{\mu p_0 | \, 0 \leq \mu < 1\} \cap \mathbb{Z}$,
- CNS can be described by using SRS.

One step in the backward division algorithm

$$T\Big(\sum_{i=0}^{n} a_i x^i\Big) = \sum_{i=0}^{n-1} a_{i+1} x^i - q \sum_{i=0}^{d-1} p_{i+1} x^i, q = \left\lfloor \frac{a_0}{p_0} \right\rfloor .$$

Representation of $\mathbb{E}$ as vector space

$\mathcal{E}$ can be represented as g-dimensional module over $\mathbb{Z}$. Let $\mathcal{B} = \{b_1, \ldots, b_g\}$ a base of it. $\mathbb{E}$ can be represented as g-dimensional vector space over $\mathbb{Q}$, also with $\mathcal{B}$ as a base. Define

$$\phi_{\mathcal{B}} : \mathbb{E} \to \mathbb{Q}^g$$

the bijection, that assigns an element of $\mathbb{E}$ its corresponding vector. We have

$$\phi_{\mathcal{B}}^{-1} : \mathbb{Q}^g \to \mathbb{E}, \mathbf{x} \mapsto < (b_1, \ldots, b_g), \mathbf{x} > .$$

$\phi_{\mathcal{B}}|_{\mathcal{E}}$ maps $\mathcal{E}$ onto $\mathbb{Z}^g$ in a bijective way.

$\phi_{\mathcal{B}}$ is homomorph with respect to the addition:
$\phi_{\mathcal{B}}(a) + \phi_{\mathcal{B}}(b) = \phi_{\mathcal{B}}(a+b)$ for $p, q \in \mathbb{E}$.

Let $\Phi_{\mathcal{B}} : \mathbb{E} \mapsto \mathbb{Q}^{g \times g}$ the embedding such that for $p, q \in \mathbb{E}$

- $\Phi_{\mathcal{B}}(p) + \Phi_{\mathcal{B}}(q) = \Phi_{\mathcal{B}}(p+q)$,
- $\Phi_{\mathcal{B}}(p)\Phi_{\mathcal{B}}(q) = \Phi_{\mathcal{B}}(pq)$,
- $\Phi_{\mathcal{B}}(p^{-1}) = \Phi_{\mathcal{B}}(p)^{-1}$,
- $\Phi_{\mathcal{B}}(p)\phi_{\mathcal{B}}(q) = \phi_{\mathcal{B}}(pq)$,
- $\det(\Phi_{\mathcal{B}}(p)) = N(p)$.

Additionally $\Phi_{\mathcal{B}}(p) \in \mathbb{Z}^{g \times g}$ for $p \in \mathcal{E}$.

Canonical digit set for general $\mathcal{E}$

We define the digit set with respect to some base $\mathcal{B}$.

$$\mathcal{N}_{\mathcal{B}}(p_0) = \{\phi_{\mathcal{B}}^{-1}\Phi_{\mathcal{B}}(p_0)(\mu_1, \ldots, \mu_g)^T | 0 \leq \mu_i < 1 \, (1 \leq i \leq g)\} \cap \mathcal{E}.$$

When we represent $\mathcal{E}$ in a base $\mathcal{B}$ then

$$\varphi_{\mathcal{B}}(\mathcal{N}_{\mathcal{B}}(p_0)) = \{\Phi_{\mathcal{B}}(p_0)(\mu_1, \ldots, \mu_g)^T | 0 \leq \mu_i < 1 \, (1 \leq i \leq g)\} \cap \mathbb{Z}^g$$

is a representation of $\mathcal{N}_{\mathcal{B}}(p_0)$ with respect to $\mathcal{B}$. $\varphi_B(\mathcal{N}_{\mathcal{B}}(p_0))$ consist of the integer points contained in the half opened g-dimensional parallelepiped induced by $\Phi_{\mathcal{B}}(p_0)$.

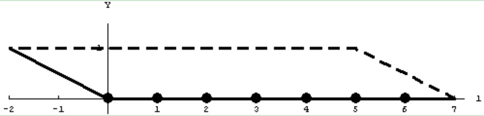
Example

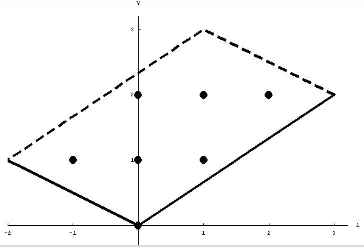
Let $\mathcal{E}$ the Eisenstein integers, thus $\mathcal{E} = \mathbb{Z}[Y]$ with $Y = e^{\frac{2\pi i}{3}}$ $(Y^2 + Y + 1 = 0)$. Suppose $p_0 = 3 + 2Y$. $N(p_0) = \det(\Phi_{\mathcal{B}}(p_0)) = 7$.

$$\mathcal{B}' = \{1 - 2Y, Y\}, \Phi_{\mathcal{B}'}(p_0) = \begin{pmatrix} 7 & -2 \\ 14 & -3 \end{pmatrix}$$

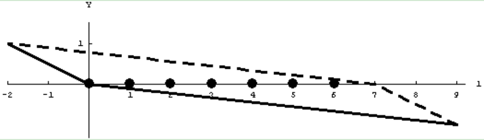
$$\mathcal{B} = \{1, Y\}$$

$$\Phi_{\mathcal{B}}(p_0) = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$$





$$\mathcal{B}'' = \{1 - 3Y, Y\}, \Phi_{\mathcal{B}''}(p_0) = \begin{pmatrix} 9 & -2 \\ 26 & -5 \end{pmatrix}$$



The backward division algorithm

With digit set $\mathcal{N}_{\mathcal{B}}(p_0)$ we have

$$T(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n-1} a_{i+1} x^i - q \sum_{i=0}^{d-1} p_{i+1} x^i$$

with

$$q = \left\lfloor \frac{a_0}{p_0} \right\rfloor_{\mathcal{B}} = \varphi_{\mathcal{B}}^{-1}\left(\left\lfloor \varphi_{\mathcal{B}}\left(\frac{a_0}{p_0}\right)\right\rfloor\right)$$

where the floor function is applied separately on each component.

$$\varphi_{\mathcal{B}}(T(\sum_{i=0}^{n} a_i x^i)) = \sum_{i=0}^{n-1} \varphi_{\mathcal{B}}(a_{i+1}) x^i - \sum_{i=0}^{d-1} \Phi_{\mathcal{B}}(p_{i+1}) \mathbf{q} x^i,$$

$$\mathbf{q} = \left\lfloor \varphi_{\mathcal{B}}(p_0^{-1} a_0) \right\rfloor .$$

Example

$\mathcal{E} = \mathbb{Z}[Y]$ with $Y = e^{\frac{2\pi i}{3}}$, $P = x^2 + Yx + 3 + 2Y$. We want the X-ary expansion of $\pi(A)$ where $A = A_0 = x(-2Y) + 3 + Y$ with digit set $\mathcal{N}_{\mathcal{B}}(3+2Y)$ for $\mathcal{B} = \{1, Y\}$.

$$q = \left\lfloor \frac{3+Y}{3+2Y} \right\rfloor_{\mathcal{B}} = \left\lfloor \frac{5-3Y}{7} \right\rfloor_{\mathcal{B}} = \varphi_{\mathcal{B}}^{-1} \left\lfloor \begin{pmatrix} \frac{5}{7} \\ -\frac{3}{7} \end{pmatrix} \right\rfloor = -Y$$

$$\Rightarrow A_1 = T(A_0) = -2Y - q(x+Y) = xY + (-3Y - 1),$$
$$e_0 = 3 + Y - q(3+2Y) = 1 + 2Y.$$

Continuing in the same manner yields

$$\begin{matrix} A_2 = x(1+Y) + (-1+Y) \\ e_1 = 0 \end{matrix}, \quad \begin{matrix} A_3 = 1+Y \\ e_2 = -1+Y \end{matrix}, \quad \begin{matrix} A_4 = 0 \\ e_3 = 1+Y \end{matrix}.$$

$$\Rightarrow \pi(A) = (1+2Y) + X^2(-1+Y) + X^3(2Y) + X^4(-1+Y) + X^5(1+Y).$$

Example

Now we calculate the X-ary expansion of $\pi(A)$, $A = A_0 = x(-2Y) + 3 + Y$ with digit set $\mathcal{N}_{\mathcal{B}'}(3+2Y)$ for $\mathcal{B}' = \{1-2Y, Y\}$.

$$q = \left\lfloor \frac{3+Y}{3+2Y} \right\rfloor_{B'} = \left\lfloor \frac{5-3Y}{7} \right\rfloor_{\mathcal{B}'} = \varphi_{\mathcal{B}'}^{-1} \left\lfloor \begin{pmatrix} \frac{5}{7} \\ 1 \end{pmatrix} \right\rfloor = Y$$

$$\Rightarrow A_1 = T(A_0) = -2Y - q(x+Y) = x(-Y) + (1-Y), \quad e_0 = 3 + Y - q(3+2Y) = 5.$$

$$\begin{matrix} A_2 = x(1-Y) + 1 + Y \\ e_1 = 6 \end{matrix}, \quad \begin{matrix} A_3 = x(-1) + 2 \\ e_2 = 3 \end{matrix}, \quad \begin{matrix} A_4 = -Y \\ e_3 = 2 \end{matrix}$$

$$\begin{matrix} A_5 = x(1-Y) + 1 + 2Y \\ e_4 = 5 \end{matrix}, \quad \begin{matrix} A_6 = x(-2Y) + 3 + Y \\ e_5 = 5 \end{matrix}.$$

Thus the backward division algorithm ends up periodically.

$$\mathbf{R} = (R_0, R_1, \ldots, R_{d-1}) \in \mathbb{R}^{g \times g \times d}$$

Each of the R_i is a $g \times g$ matrix.

$$\begin{aligned}\omega_{\mathbf{R}} :& \mathbb{Z}^{g \times d} \to \mathbb{Z}^{g \times d},\\ & (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d-1}) \mapsto (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{d-1}, -\lfloor \mathbf{R}\mathbf{x} \rfloor),\end{aligned}$$

where $\mathbf{R}\mathbf{x} = \sum_{i=0}^{d-1} R_i \mathbf{x}_i$.

Definition

ω is called a square shift radix system (2SRS) if for all $\mathbf{x} \in \mathbb{Z}^{g \times d}$ there exists a $k \in \mathbb{N}$ such that $\omega_{\mathbf{R}}^k(\mathbf{x}) = \mathbf{0}$.

$$\mathcal{T}_{g,d} := \left\{ \mathbf{R} \in \mathbb{R}^{g \times g \times d} \middle| \omega_{\mathbf{R}} \text{ is ultimately periodic } \forall \mathbf{x} \in \mathbb{Z}^{g \times d} \right\}$$

$$\mathcal{T}_{g,d}^0 := \left\{ \mathbf{R} \in \mathbb{R}^{g \times g \times d} \middle| \omega_{\mathbf{R}} \text{ is a 2SRS} \right\}.$$

Basic properties of square shift radix systems

Lemma

Let $\mathbf{R} := (R_0, \dots, R_{d-1}) \in \mathbb{R}^{g\times g\times d}$ *and*
$Q(\lambda) = \det(\lambda^d I_d + \lambda^{d-1} R_{d-1} + \cdots + \lambda R_1 + R_0) \in \mathbb{R}[\lambda]$
$(\deg(Q(\lambda)) = gd)$.

- *If* $\mathbf{R} \in \mathcal{T}_{g,d}$ *then all roots of* Q *have modulus smaller or equal* 1,
- *If all roots of* Q *have modulus smaller* 1 *then* $\mathbf{R} \in \mathcal{T}_{g,d}$.

Lemma

$\mathcal{D}_d = \mathcal{T}_{1,d}$ *and* $\mathcal{D}_d^0 = \mathcal{T}_{1,d}^0$.

Theorem

For $\mathbf{R} \in \mathrm{int}(\mathcal{T}_{g,d})$ *there exists an algorithm to verifies whether* $\mathbf{R} \in \mathcal{T}_{g,d}^0$.

Representation of $K(\mathcal{R})$

Let

$$w_0 := p_d, w_k := Xw_{k-1} + p_{d-k} (1 \leq k < d).$$

$\{w_0, \ldots, w_{d-1}\}$ is a base of $K(\mathcal{R})$.

$$K(\mathcal{R}) = \{\sum_{i=0}^{d-1} w_i a_i | a_i \in \mathcal{E}\}.$$

Theorem

Let $\mathcal{B}$ a base of $\mathcal{E}$, $\mathbf{R} = (\Phi_\mathcal{B}(p_d)\Phi_\mathcal{B}(p_0^{-1}), \ldots, \Phi_\mathcal{B}(p_1)\Phi_\mathcal{B}(p_0^{-1}))$, $a = \sum_{i=0}^{d-1} w_i a_i \in K(\mathcal{R})$ and $T(a) = \sum_{i=0}^{d-1} w_i a'_i$ with digit set $\mathcal{N}_\mathcal{B}(p_0)$. Then we have

$$(\varphi_\mathcal{B}(a'_0), \ldots, \varphi_\mathcal{B}(a'_{d-1})) = \omega_\mathbf{R}(\varphi_\mathcal{B}(a_0), \ldots, \varphi_\mathcal{B}(a_{d-1})).$$

Theorem

Let $\mathcal{B}$ a base of $\mathcal{E}$. $(P, \mathcal{N}_{\mathcal{B}}(p_0))$ is a digit set if and only if

$$(\Phi_{\mathcal{B}}(p_d p_0^{-1}), \dots, \Phi_{\mathcal{B}}(p_1 p_0^{-1})) \in \mathcal{T}_{g,d}^0.$$

Example

Let $\mathcal{E} = \mathbb{Z}[Y]$ with $Y^2 + Y + 1 = 0$ and $P = x^2 + Yx + 3 + 2Y$.

- Let $B = \{1, Y\}$.

$$(\Phi_{\mathcal{B}}(p_0^{-1}), \Phi_{\mathcal{B}}(p_1 p_0^{-1})) = \left(\begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}, \begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{pmatrix} \right) \in \mathcal{T}_{2,2}^0.$$

  Thus $(P, \mathcal{N}_{\mathcal{B}})$ is a digit system.

- For $\mathcal{B}' = \{1 - 2Y, Y\}$ we already know that $(P, \mathcal{N}_{\mathcal{B}'})$ is no digit system.

Problems and Thanks

- For which p_0 there exists a base $\mathcal{B}$ such that $\mathcal{N}_{\mathcal{B}}(p_0) \subset \mathbb{Z}$?
- Characterise $\mathcal{T}_{g,d}$ or even $\mathcal{T}_{g,d}^0$.
- We know that SRS can be used to describe the β-expansion. Which dynamical systems can be described by 2SRS?

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The slides are (soon) available : www.palovsky.com
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