

REAL SETS INDUCED BY ENDOMORPHISMS OF THE FREE MONOID

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Definitions and Notations

$\mathcal{A} = \{1, 2, \dots, m\}$	finite alphabet
$\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$	free monoid over \mathcal{A} consisting of the set of non-empty words \mathcal{A}^+ and the empty word ε (neutral element)
$\overline{\mathcal{A}} = \{\bar{1}, \bar{2}, \dots, \bar{m}\}$	set of inverse letters
$\overline{\mathcal{A}}^* := \overline{\mathcal{A}}^+ \cup \{\varepsilon\}$	free monoid over $\overline{\mathcal{A}}$
$(\mathcal{A} \cup \overline{\mathcal{A}})^*$	finite words over $\mathcal{A} \cup \overline{\mathcal{A}}$
\sim	equivalence relation on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the cancellation law $x\bar{x} \sim \varepsilon \sim \bar{x}x$ (for $x \in \mathcal{A}$)

Observe that $(\mathcal{A} \cup \overline{\mathcal{A}})^* / \sim$ is the free group over \mathcal{A} . For $X = x_1 x_2 \dots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ the word $\bar{X} := \bar{x}_n \dots \bar{x}_1 \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ is the inverse of X (modulo \sim) where $\bar{\bar{x}} = x$ for all $x \in \mathcal{A}$.

Definitions and Notations

For $X = x_1x_2 \dots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ and $y \in \mathcal{A}$ define

$$|X|_y = \# \{j \in \{1, \dots, n\} : x_j = y\} - \# \{j \in \{1, \dots, n\} : x_j = \bar{y}\}$$

$$|X| = \sum_{y \in \mathcal{A}} |X|_y$$

$$\mathbf{I}(X) = (|X|_1, |X|_2, \dots, |X|_m).$$

We denote by \preceq the partial ordering on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the prefix-condition:

$$X \preceq Y \iff \overline{X}Y \in \mathcal{A}^* \pmod{\sim},$$

$$X \prec Y \iff \overline{X}Y \in \mathcal{A}^+ \pmod{\sim}.$$

Note that for $A, B \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $A \prec B$ the set

$$\{X \in (\mathcal{A} \cup \overline{\mathcal{A}})^* / \sim : A \preceq X \preceq B\}$$

is a \prec -chain.

Morphisms of the free monoid

Denote by $\sigma : \mathcal{A}^* \mapsto \mathcal{A}^*$ a primitive morphism (substitution), hence, there exists a positive integer n such that $|\sigma^n(y)|_x \geq 1$ for all $x, y \in \mathcal{A}$.

We define

$\mathbf{M} := (|\sigma(y)|_x)_{1 \leq x, y \leq m}$ the incidence matrix of σ ;
 $\theta > 1$ the dominant Perron-Frobenius eigenvalue of \mathbf{M} ;

$\mathbf{v} = (v_1, \dots, v_m)$ a strictly positive left eigenvector of \mathbf{M} with respect to θ .

We can extend σ easily to $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ by defining $\sigma(\bar{x}) = \overline{\sigma(x)}$ for all $x \in \mathcal{A}$.

Coding prescriptions

Definition

A *coding prescription* (with respect to σ) is a function c with domain \mathcal{A}^2 that assigns to each pair of letters a finite set of integers such that

1. for each $x \in \mathcal{A}$ the set $c(xx)$ is a complete residue system modulo $|\sigma(x)|$ such that for all $k \in c(xx)$ we have $-|\sigma(x)| < k < |\sigma(x)|$;
2. for all $ab \in \mathcal{A}^2$ we have
$$c(ab) = \{k \in c(aa) : k < 0\} \cup \{0\} \cup \{k \in c(bb) : k > 0\}.$$

We call a coding prescription *continuous* if $c(ab)$ consists of consecutive integers for each $ab \in \mathcal{A}^2$.

Graph associated to a coding prescription

We associate to a coding prescription a finite graph $H_{\sigma,c}$ with vertex set \mathcal{A}^2 and an edge from ab to a_1b_1 labelled by (D, a_1b_1) with $D \in \mathcal{A}^* \cup \overline{\mathcal{A}}^*$ whenever $|D| \in c(ab)$ and $\overline{\sigma(a)} \preceq D\bar{a}_1 \prec D \prec Db_1 \preceq \sigma(b)$.

Observe that each vertex ab has exactly $|c(ab)|$ outgoing edges.

For each positive integer n we denote by $H_{\sigma,c}^n(ab)$ the set of paths of length n that start in ab . Analogously $H_{\sigma,c}^\infty(ab)$ is the set of infinite walks that start in ab .

Theorem 1

Let n be a positive integers and define the function $c^{(n)}$ on \mathcal{A}^2 by

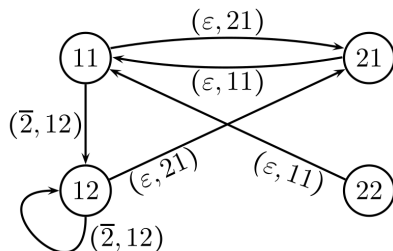
$$c^{(n)}(ab) := \left\{ \sum_{j=1}^n |\sigma^{n-j}(D_j)| : (D_j, a_j b_j)_{j=1}^n \in H_{\sigma, c}^n(ab) \right\} \subset \mathbb{Z}.$$

Then $c^{(n)}$ is a coding prescription with respect to σ^n .

Example (Fibonacci substitution)

$$\begin{aligned}\sigma(1) &= 12, \\ \sigma(2) &= 1.\end{aligned}$$

$$\begin{aligned}c(11) = c(12) &= \{-1, 0\}, \\ c(21) = c(22) &= \{0\}.\end{aligned}$$



We have $\sigma^2 : 1 \mapsto 121, 2 \mapsto 12$ and $\sigma(\bar{2}) = \bar{1}$, therefore

$$c^{(2)}(11) = c^{(2)}(12) = \{-2, -1, 0\}, c^{(2)}(21) = c^{(2)}(22) = \{-1, 0\}.$$

Infinite paths

For each $ab \in \mathcal{A}^2$ define

$$I_{ab} := \left\{ \sum_{j \geq 1} \langle \mathbf{l}(D_j), \mathbf{v} \rangle \theta^{-j} : (D_j, a_j b_j)_{j \geq 1} \in H_{\sigma, c}^{\infty}(ab) \right\} \subset \mathbb{R}.$$

Note that always $0 \subseteq I_{ab}$ where equality may hold.

Problem

Characterise the sets I_{ab} (in terms of c and σ).

Some fundamental properties

Lemma 2

There exists a positive constant $K \in \mathbb{R}$ such that for each $ab \in \mathcal{A}^2$

$$I_{ab} = K \cdot \lim_{n \rightarrow \infty} \theta^{-n} c^{(n)}(ab)$$

(the limit with respect to the Hausdorff metric).

Proposition 3

For each $ab \in \mathcal{A}^2$

1. I_{ab} is a compact set;
2. $I_{ab} \subseteq [-v_a, v_b]$;
3. $I_{ab} = (I_{aa} \cap (-\infty, 0)) \cup \{0\} \cup (I_{bb} \cap (0, \infty))$;
- 4.

$$I_{ab} = \bigcup_{(D, a_1 b_1) \in H_{\sigma, c}^1(ab)} \theta^{-1}(\langle I(D), \mathbf{v} \rangle + I_{a_1 b_1}).$$

Continuous coding prescriptions

Proposition 4

Suppose that c is continuous.

- ▶ For each $ab \in \mathcal{A}^2$ we have $I_{ab} = [-v_a^-, v_b^+]$ with $v_a^-, v_b^+ \geq 0$.
- ▶ $(v_1^-, \dots, v_m^-) + (v_1^+, \dots, v_m^+) = (v_1, \dots, v_m)$.
- ▶ For each $ab \in \mathcal{A}^2$ the particular subsets in the union

$$I_{ab} = \bigcup_{(D, a_1 b_1) \in H_{\sigma, c}^1(ab)} \theta^{-1}(\langle l(D), \mathbf{v} \rangle + I_{a_1 b_1})$$

have pairwise disjoint interior.

A special case

Proposition 5

Suppose that $|\sigma(x)| \equiv 1 \pmod{2}$ for all $x \in \mathcal{A}$ and $c(ab) \subset 2\mathbb{Z}$ for all $ab \in \mathcal{A}^2$. Then the following items hold for all $ab \in \mathcal{A}^2$.

- ▶ $I_{ab} = [-v_a, v_b]$.
- ▶ The particular subsets in the union

$$I_{ab} = \bigcup_{(D, a_1 b_1) \in H_{\sigma, c}^1(ab)} \theta^{-1}(\langle \mathbf{l}(D), \mathbf{v} \rangle + I_{a_1 b_1})$$

have pairwise disjoint interior.

Questions

Question 1

For which setting is I_{ab} an interval?

Question 2

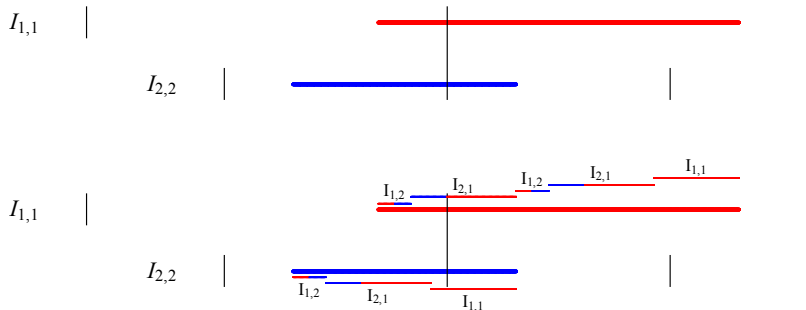
For which setting do the subsets in the union (4. of Proposition 3) have disjoint interior? Alternatively, which setting corresponds to a graph directed iterated function system?

Question 3

What is the Lebesgue measure of I_{ab} ?

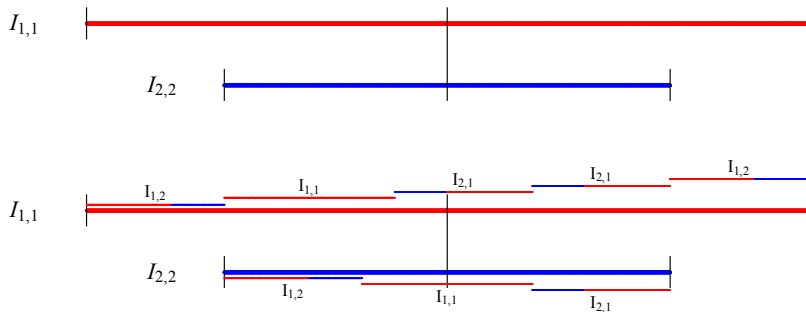
Example 1

$$\begin{aligned}\sigma : \quad 1 &\mapsto 12112, & 2 &\mapsto 121 \\ c : \quad 11 &\mapsto \{-1, 0, 1, 2, 3\}, & 22 &\mapsto \{-2, 1, 0\}\end{aligned}$$



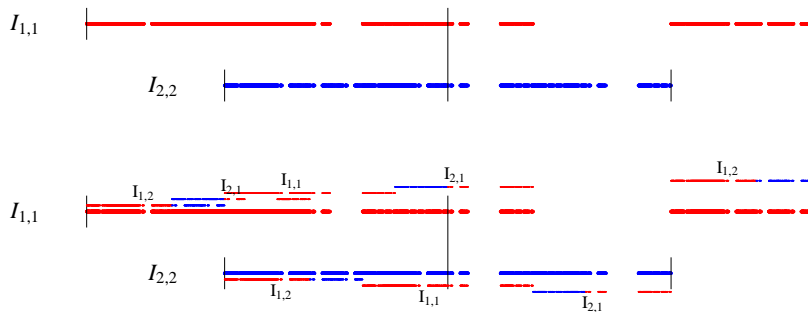
Example 2

$$\begin{aligned}\sigma : \quad 1 &\mapsto 12112, & 2 &\mapsto 121 \\ c : \quad 11 &\mapsto \{-4, -2, 0, 2, 4\}, & 22 &\mapsto \{-2, 0, 2\}\end{aligned}$$



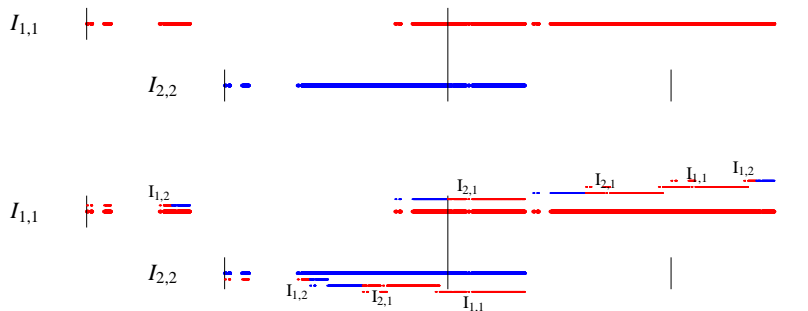
Example 3

$$\begin{aligned} \sigma : \quad 1 &\mapsto 12112, & 2 &\mapsto 121 \\ c : \quad 11 &\mapsto \{-4, -3, -2, 0, 4\}, & 22 &\mapsto \{-2, 0, 2\} \end{aligned}$$



Example 4

$$\begin{aligned}\sigma : \quad 1 &\mapsto 12112, & 2 &\mapsto 121 \\ c : \quad 11 &\mapsto \{-4, -3, -2, 0, 4\}, & 22 &\mapsto \{-2, -1, 0\}\end{aligned}$$



Example 5

$$\begin{aligned}\sigma : \quad 1 &\mapsto 121, & 2 &\mapsto 12 \\ c : \quad 11 &\mapsto \{-2, -1, 0\}, & 22 &\mapsto \{0, 1\}\end{aligned}$$

