Three dimensional symmetric Shift Radix Systems

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Strobl, July 2007

Supported by FWF, project S9610-N13

Definition

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For $\mathbf{r} \in \mathbb{R}^d$, $0 \leq \epsilon \leq \frac{1}{2}$ let

$$\tau_{\mathbf{r},\varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{rx} + \varepsilon \rfloor).$$

$\tau_{\mathbf{r},\varepsilon}$ is called an ε-shift radix system (ε-SRS) if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r},\varepsilon}^n(\mathbf{x}) = \mathbf{0}.$$

- $\varepsilon = 0$: Original Shift Radix Systems by Akiyama *et al* (2005).
- $\varepsilon = \frac{1}{2}$: Symmetric shift radix systems (SSRS) by Akiyama and Scheicher

The sets $\mathcal{D}_d(\varepsilon)$ and $\mathcal{D}_d^0(\varepsilon)$

$$\begin{aligned}
\mathcal{D}_d(\varepsilon) &:= \{\mathbf{r} \in \mathbb{R}^d | \forall \mathbf{x} \in \mathbb{Z}^d \exists n, l \in \mathbb{N} : \tau_{\mathbf{r},\varepsilon}^k(\mathbf{x}) = \tau_{\mathbf{r},\varepsilon}^{k+l}(\mathbf{x}) \ \forall k \geq n\} \\
\mathcal{D}_d^0(\varepsilon) &:= \{\mathbf{r} \in \mathbb{R}^d | \tau_{\mathbf{r},\varepsilon} \text{ is an } \varepsilon\text{-SRS}\}
\end{aligned}$$

For $\mathbf{r} = (r_0, \dots, r_{d-1})$ denote by $R(\mathbf{r}) \in \mathbb{R}^{d \times d}$ the companion matrix with characteristic polynomial $x^d + r_{d-1}x^{d-1} + \cdots + r_0$. We have

- $\mathcal{D}_d^0(\varepsilon) \subset \mathcal{D}_d(\varepsilon)$,
- $\mathbf{r} \in \mathrm{int}\mathcal{D}_d(\varepsilon) \Leftrightarrow R(\mathbf{r})$ has spectralradius < 1,
- $\mathcal{D}_d(\varepsilon) \cap (X_1 = 0) = \mathcal{D}_{d-1}(\varepsilon)$,
  $\mathcal{D}_d^0(\varepsilon) \cap (X_1 = 0) = \mathcal{D}_{d-1}^0(\varepsilon)$.
- $\mathrm{int}(\mathcal{D}_d(\varepsilon_1)) = \mathrm{int}(\mathcal{D}_d(\varepsilon_2))$ but in general $\partial\mathcal{D}_d(\varepsilon_1) \neq \partial\mathcal{D}_d(\varepsilon_2)$ for $\varepsilon_1, \varepsilon_2 \in [0, \frac{1}{2}]$, $\varepsilon_1 \neq \varepsilon_2$.

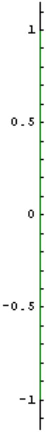


Figure: $\mathcal{D}_1(\varepsilon)$

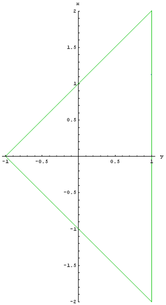


Figure: $\mathcal{D}_2(\varepsilon)$

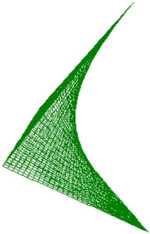


Figure: $\mathcal{D}_3(\varepsilon)$

Canonical Number Systems

Definition

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$ an expanding polynomial, $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ and $x = X(P(X)\mathbb{Z}[X]) \in R$. If every $A(x) \in R$ can be written in the form

$$A(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathcal{N} := [0, |p_0|) \cap \mathbb{Z},$$

then $(P(X), \mathcal{N})$ is called a Canonical Number System (CNS).

Theorem (Akiyama *et al*)

$(P(X), \mathcal{N})$ is a CNS if and only if $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0(0)$.

Generalised Canonical Number Systems

Definition

Let $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$ an expanding polynomial, $\varepsilon \in [0, \frac{1}{2}]$, $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$ and $x = X(P(X)\mathbb{Z}[X]) \in R$. If every $A(x) \in R$ can be written in the form

$$A(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathcal{N} := [-\varepsilon|p_0|, (1-\varepsilon)|p_0|) \cap \mathbb{Z},$$

then $(P(X), \mathcal{N})$ is called an ε-Canonical Number System (ε-CNS).

Theorem

$(P(X), \mathcal{N})$ is a ε-CNS if and only if $(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0(\varepsilon)$.

β-expansion

Definition

Let $\beta > 1$ a Pisot number with minimal Polynomial $A(X) = X^{d+1} + a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$. β is said to have property (F) if each $\gamma \in \mathbb{Z}[\frac{1}{\beta}]$ has a unique representation of the shape

$$\gamma = \sum_{i=m}^{n} a_i \beta^i, \quad a_i \in \mathcal{A} := [0, \beta) \cap \mathbb{Z}$$

with $\sum_{i=m}^{j} a_i \beta^i \in \beta^{j+1}[0,1) \quad (\forall n \leq j \leq m)$.

Theorem (Akiyama *et al*)

Let $r_{d-1} = p_d + \beta, r_{d-2} = p_{d-1} + \beta r_{d-1}, \ldots, r_0 = p_1 + \beta r_1$. β has property (F) if and only if $(r_0, r_1, \ldots, r_{d-1}) \in \mathcal{D}_d^0(0)$.

Generalised β-expansion

Definition

Let $\beta > 1$ a Pisot number with minimal Polynomial $A(X) = X^{d+1} + a_d X^d + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$. β is said to have property (F-ε) if each $\gamma \in \mathbb{Z}[\frac{1}{\beta}]$ has a unique representation of the shape

$$\gamma = \sum_{i=m}^{n} a_i \beta^i, \quad a_i \in \mathcal{A} := (-(\beta-1)\varepsilon - 1, (\beta-1)(1-\varepsilon)+1) \cap \mathbb{Z}$$

with $\sum_{i=m}^{j} a_i \beta^i \in \beta^{j+1}[-\varepsilon, 1-\varepsilon) \quad (\forall n \leq j \leq m)$.

Theorem

Let $r_{d-1} = p_d + \beta, r_{d-2} = p_{d-1} + \beta r_{d-1}, \ldots, r_0 = p_1 + \beta r_1$. β has property (F-ε) if and only if $(r_0, r_1, \ldots, r_{d-1}) \in \mathcal{D}_d^0(\varepsilon)$.

Periods

Definition

For a given $\mathbf{r} \in \mathcal{D}_d(\varepsilon)$ we call $(x_0, \ldots, x_{l-1}) \in \mathbb{Z}^l$ a period of $\tau_{\mathbf{r},\varepsilon}$ if for all $0 \leq i < l$

$$\tau_{\mathbf{r},\varepsilon}\left((x_i, \ldots, x_{d+i-1})\right) = (x_{i+1}, \ldots, x_{i+d})$$

(indices modulo l).

Lemma

For $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d(\varepsilon))$ there are only finitely many periods of $\tau_{\mathbf{r},\varepsilon}$ and $\mathbf{r} \in \mathcal{D}_d^0(\varepsilon)$ if and only if (0) is the only period of $\tau_{\mathbf{r},\varepsilon}$.

Periods

Lemma

Let $\pi = (x_0, \dots, x_{l-1}) \in \mathbb{Z}^l$ and $P(\pi) \subset \mathbb{R}^d$ all points $(s_0, \dots, s_{d-1})$ that satisfy the system of l double inequalities

$$0 \le s_0 x_i + \cdots + s_{d-1} x_{d+i-1} + x_{i+d} + \varepsilon < 1$$

for $0, \le i < l$ (indices of x modulo l). Then π is a period of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in P(\pi)$.

It is much easier to show an area $Q \subset \mathcal{D}_d(\varepsilon)$ not to belong to $\mathcal{D}_d^0(\varepsilon)$ than to show that it is part of $\mathcal{D}_d^0(\varepsilon)$.

Algorithmic way

$D_d^0(\varepsilon)$ can be gained by cutting out polyhedra from $D_d(\varepsilon)$.

Theorem (Brunotte)

For a closed convex set $Q \in \mathrm{int}(\mathcal{D}_d(\varepsilon))$ there exists an algorithm that returns a finite set Π of integer vectors such that

$$Q \setminus \bigcup_{\pi \in \Pi} P(\pi) = Q \cap \mathcal{D}_d^0(\varepsilon).$$

Corollary

Let $\varepsilon \in [0, \frac{1}{2}]$. If there exists a closed set $\tilde{D} \subset \mathrm{int}(D_d(\varepsilon))$ with $D_d^0(\varepsilon) \subset \tilde{D}$ than $D_d^0(\varepsilon)$ can be fully characterised by cutting out finitely many polyhedra from $D_d(\varepsilon)$.

$\mathcal{D}_2^0(0)$

$\mathcal{D}_2^0(0)$ cannot be characterised by cutting out only finitely many polyhedra.

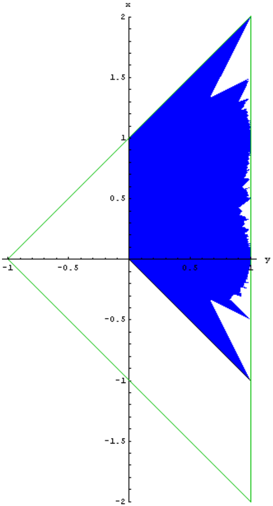


Figure: $\mathcal{D}_2^0(0)$ (approx.)

$\mathcal{D}_2^0(\frac{1}{2})$

Theorem (Akiyama, Scheicher)

$\mathcal{D}_2^0(\frac{1}{2})$ *can be characterised by cutting out* 9 *Polyhedra.*

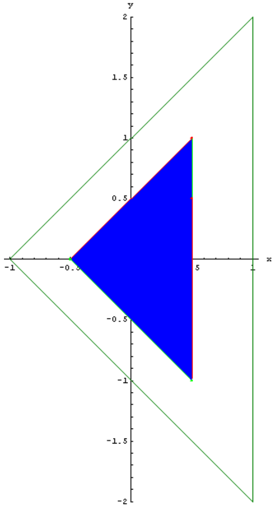


Figure: $\mathcal{D}_2^0(\frac{1}{2})$

$\mathcal{D}_2^0(\varepsilon)$ for $\varepsilon \in (0, \frac{1}{2})$

Theorem

For $\varepsilon \in (0, \frac{1}{2})$ the set $\mathcal{D}_2^0(\varepsilon)$ is completely contained in a closed set $\tilde{D} \subset \mathrm{int}(\mathcal{D}_d(\varepsilon))$. Therefore $\mathcal{D}_2^0(\varepsilon)$ can be fully characterised by cutting out finitely many polyhedra from $\mathcal{D}_d(\varepsilon)$. We have

$$\inf_{\mathbf{x} \in \mathcal{D}_2^0(\varepsilon), \mathbf{y} \in \partial \mathcal{D}_2(\varepsilon)} \|\mathbf{x} - \mathbf{y}\|_\infty \leq \epsilon.$$

$\mathcal{D}_3^0(\frac{1}{2})$

Theorem (Huszti, Scheicher, Thuswaldner, Surer)

$\mathcal{D}_2^0(\frac{1}{2})$ can be characterised by cutting out 43 Polyhedra.

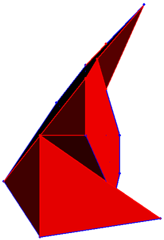


Figure: $\mathcal{D}_3^0(\frac{1}{2})$

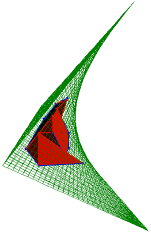
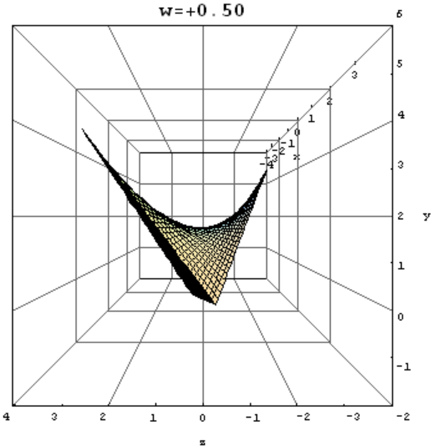


Figure: Position of $\mathcal{D}_3^0(\frac{1}{2})$ inside $\mathcal{D}_3(\frac{1}{2})$

Questions

- Is it possible to gain further information about $\mathcal{D}_d^0(0)$ by looking at $\mathcal{D}_d^0(\varepsilon)$ when $\varepsilon \to 0$.
- For $d = 1, 2$ we have $\mathcal{D}_d^0(0) \cap \partial\mathcal{D}_d(0) = \emptyset$. Is this true for $d \geq 3$?
- Can $\mathcal{D}_d^0(\varepsilon)$ with $0 < \varepsilon < \frac{1}{2}$ be completely characterised for $d \geq 3$?
- Can $\mathcal{D}_d^0(\frac{1}{2})$ be completely characterised for $d \geq 4$?

Thanks



The research was supported by the FWF, project S9610-N13

The slides are (soon) available : www.palovsky.com

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