A VERY GENERAL APPROACH FOR REPRESENTING INTEGERS

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In the presentation we are interested in different notions of integer numeration. Apart from the well-known q-ary expansions, the Zeckendorf representation [6] is quite famous. It is based on the Fibonacci sequence $(F_j)_{j\geq 0}$ where $F_0 := 0$, $F_1 := 1$ and $F_j := F_{j-1} + F_{j-2}$ for $j \geq 2$ and allows us to uniquely represent each positive integer as a sum of Fibonacci numbers of index larger than 1 that does not contain two consecutively indexed ones.

This concept also works for other linear recurrences which yields the so-called the G-ary expansions (see [4]). For examples, define $(G_k)_{k\geq 1}$ by $G_0 := 1$, $G_1 := 5$, $G_k := 4G_{k-1} + 3G_{k-2}$. Then each positive integer x can be uniquely represented as finite sum

$$x = \sum_{j=1}^{n} d_j G_{n-j}$$

where each G-ary digit d_j is contained in the set $\{0, \ldots, 4\}$, $d_1 \neq 0$ and $d_j, d_{j+1} \leq_{\text{lex}} 4, 2$, that is two consecutive digits are lexicographically smaller than or equal to $4, 2$.

A very general approach for representing integers was presented by Dumont and Thomas [1]. It is based on non-erasing morphisms of the free monoid (so-called substitutions) and covers representations with respect to linear recurrence sequences (see [2]). For example, the Zeckendorf representation corresponds to the famous Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$ (over the alphabet $\{1, 2\}$) while our other example can be recovered via the substitution $1 \mapsto 11112, 2 \mapsto 111$. The digits in the concept of Dumont and Thomas are induced by the proper prefixes of the images of the letters. In the case of the latter substitution the prefixes of the image of 1 are $1, 11, 111, 1111$ and the prefixes of the image of 2 are $1, 11$. The set of G-ary digits is given by the length of the prefixes (and 0).

More recently Knuth [3] presented the negaFibonacci expansions that also involves negative integers. We extend the Fibonacci sequence and define for each $j \in \mathbb{Z}$ the Fibonacci number F_j by $F_j := F_{j-1} + F_{j-2}$ (with $F_0 := 0$, $F_1 := 1$). This yields $F_{-j} = (-1)^{j-1} F_j$ for all $j \in \mathbb{Z}$. The Fibonacci numbers with negative indices $F_{-1} = 1, F_{-2} = -1, F_{-3} = 2, \ldots$ are called the negaFibonacci numbers. Quite analogously to the Zeckendorf representation we can represent each (positive and negative) integer uniquely as a sum of pairwise not consecutively indexed negaFibonacci numbers. Obviously the negaFibonacci expansion does not fit into the concept of Dumont and Thomas.

The main purpose of the talk is to present a very recent research [5] that generalises the idea of Dumont and Thomas. It also takes suffixes into consideration that result in negative digits. More precisely, for each letter we choose a set of prefixes and suffixes where we have a lot of freedoms for our choice. Therefore, we can associate to one substitution a large amount of integer numeration systems that all have different behaviour and also different domains of representable integers. We will see that within this framework we can describe the negaFibonacci expansion by a substitution. But we also will be able to obtain new variants of G-ary expansions. For instance, consider the substitution $1 \mapsto 11121, 2 \mapsto 111$ and choose for the image of 1 the the prefixes $1, 11, 111$ and the suffix 1. For the image of 2 we stay with the prefixes $1, 11$ from above without any suffixes. In this way we get a numeration system that allows us to uniquely represent each integer x as

$$x = \sum_{j=1}^{n} d_j G_{n-j} \text{ with } d_1 \neq 0,\, d_j \in \{-1, \dots, 3\},\, -1, 0 \leq_{\text{lex}} d_j, d_{j+1} \leq_{\text{lex}} 3, 2.$$

However, observe that the choice of prefixes and suffixes also has to satisfy several conditions. We will explain which rules have to be obeyed, discuss the effect of the choice on the domain of representable integers, and show what further properties the representations have.

References

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