MEASURE-WISE DISJOINT RAUZY FRACTALS WITH THE SAME INCIDENCE MATRIX

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Abstract. The properties of the intersection of Rauzy fractals associated with substitutions having the same incidence matrix have been studied by several authors. Different techniques have been introduced and used for this purpose, one of them is the balanced pair algorithm. In the present paper we explore the actual limitations of this algorithm. We show that the balanced pair algorithm is defined and terminates only when the intersection of the Rauzy fractals have non-empty interior and when this condition fails it cannot be used.

1. Introduction

Rauzy fractals play a fundamental role in the study of the long standing Pisot conjecture, which states that the dynamical system associated with an irreducible Pisot substitution has a pure point spectrum (*cf.* [17, 26]). Recently important progress in solving this conjecture has been made (*cf.* [6]). The topological, geometrical and dynamical properties of Rauzy fractals have been analysed extensively, for instance see the survey [10] and references therein.

The study of the common dynamics of two Pisot substitutions is related to the intersection of their respective Rauzy fractals, especially when the substitutions have the same incidence matrix. This problem was discussed the first time in [22] and has been subsequently studied in [18, 19, 21]. In [18] the balanced pair algorithm for two substitutions was presented. This algorithm is based on the classical balanced pair algorithm introduced by Livshits [14] in relation to the Pisot conjecture (see also [23]). It allows to describe the intersection of two Rauzy fractals by a new substitution. The conditions on the substitutions are quite strong. In particular, one Rauzy fractal is required to have the *geometric property (F)*, i.e., to contain the origin as inner point. This immediately implies that the intersection of the Rauzy fractals has non-empty interior. Furthermore, only irreducible substitutions are considered.

In the present article we generalise the results presented in [18] by removing the requirement of the geometric property (F). Additionally, we will also allow reducible substitutions as long as the Rauzy fractals induce a proper lattice tiling. We also obtain a necessary condition, namely, that the balanced pair algorithm is defined and terminates if the intersection of the corresponding Rauzy fractals has positive Lebesgue measure (Theorem 1).

Our results also have consequences on the characterisation of substitutions that have the geometric property (F). We show that a substitution that possesses this property always satisfies a certain combinatoric condition stated in terms of a periodic point. Furthermore, we give necessary conditions, based only on the spectral properties of the incidence matrix of

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the substitution, that ensure that the substitution does not have the geometric property (F) (Theorem 2).

The article is organised as follows: In Section 2 we introduce all necessary definitions and formalism, present the balanced pair algorithm, recall the results from [18] and state our main result, which will be proved in Section 3. In Section 4 we consider our results from the point of view of the geometric property (F) and give some illustrative examples.

2. Preliminaries and main result

2.1. **Words and substitutions.** Let $\mathcal{A}$ be a finite set that we call an *alphabet* and its elements *letters*. The set of finite words over $\mathcal{A}$ is given by $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$ where $\mathcal{A}^0$ is the set formed by the empty word ε.

For a word $V \in \mathcal{A}^*$ and a letter $a \in \mathcal{A}$ we denote the length of V by $|V|$ and the number of occurrences of the letter a in V by $|V|_a$, i.e., $|V| = \sum_{a\in\mathcal{A}} |V|_a$.

Assuming that $\mathcal{A} = \{1, \ldots, m\}$, we set

$$\mathbf{l}(V) := (|V|_1, |V|_2, \ldots, |V|_m) \in \mathbb{N}^m, \tag{1}$$

some authors call this map the *abelianasation map*.

A *substitution or morphism* is a map from $\mathcal{A}$ to $\mathcal{A}^*$, the substitution ζ is extended to $\mathcal{A}^*$ by concatenation, i.e., $\zeta(UV) = \zeta(U)\zeta(V)$ for $U, V \in \mathcal{A}^*$. The *incidence matrix* of ζ is the matrix $\mathbf{M}_\zeta = (|\zeta(j)|_i)_{i,j\in\mathcal{A}}$. We say the substitution ζ is *primitive* if the matrix $\mathbf{M}_\zeta$ is primitive, i.e. there exists a k such that all entries of $\mathbf{M}_\zeta^k$ are positive. Observe that for all $A \in \mathcal{A}^*$ we have $\mathbf{M}_\zeta \cdot \mathbf{l}(A) = \mathbf{l}(\zeta(A))$.

Let $\mathcal{A}^\mathbb{N}$ denote the set of one-sided infinite sequences on $\mathcal{A}$. We define the *shift map* $\sigma : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ by

$$\sigma(v_0 v_1 v_2 \ldots) := v_1 v_2 \ldots.$$

The substitution ζ can be extended in a straightforward way to $\mathcal{A}^\mathbb{N}$. A periodic point of ζ is a sequence $\mathbf{u} \in \mathcal{A}^\mathbb{N}$ such that $\zeta^k(\mathbf{u}) = \mathbf{u}$ for some positive integer k. If $k = 1$ then we call $\mathbf{u}$ a fixed point. According to [16] each substitution possesses at least one periodic point.

A primitive substitution ζ induces the substitution dynamical system (Ω_ζ, σ), where

$$\Omega_\zeta := \overline{\{\sigma^m(\mathbf{u}) \,:\, m \geq 0\}},$$

with $\mathbf{u}$ a periodic point of ζ and the closure with respect to the product topology of the discrete topology. Note that Ω_ζ does not depend on the actual choice of $\mathbf{u}$ since we require ζ to be primitive. Furthermore, the primitivity implies that the dynamical system is minimal, i.e. every orbit is dense.

2.2. **Rauzy fractals.** If ζ is a primitive substitution then the incidence matrix $\mathbf{M}_\zeta$ possesses a dominant Perron-Frobenius eigenvalue that we denote by β. Throughout the entire article we assume ζ to be a *unimodular Pisot substitution (or unit Pisot substitution)*, meaning that the dominant root β is a real algebraic unit greater than 1 and the Galois conjugates different from β have modulus less than one. We say that ζ is *irreducible* if the characteristic polynomial of the matrix $\mathbf{M}_\zeta$ is irreducible; in this case the algebraic degree of β is equal to m, the size of the alphabet $\mathcal{A}$.

Let $d+1$ denote the algebraic degree of β and $\alpha_1, \ldots, \alpha_d$ the Galois conjugates different from β. Observe that $\alpha_1, \ldots, \alpha_d$ are located inside the complex unit circle. Denote by $\mathbb{K} \cong \mathbb{R}^d$

the contractive space spanned by the right eigenvectors of $\mathbf{M}_\zeta$ associated with $\alpha_1, \dots, \alpha_d$. The (one-dimensional) subspace spanned by the right eigenvector of $\mathbf{M}_\zeta$ associated to β is the expanding space and we will denote it by $\mathbb{E}$. If ζ is irreducible then $d = m-1$ and $\mathbb{R}^m \cong \mathbb{K} \oplus \mathbb{E}$. Otherwise there is a complementary space $\mathbb{X}$ of dimension $m-d-1$ spanned by the eigenvectors associated with the remaining eigenvalues and we have $\mathbb{R}^m \cong \mathbb{K} \oplus \mathbb{E} \oplus \mathbb{X}$.

We denote by $\pi : \mathbb{R}^m \longrightarrow \mathbb{K}$ the projection onto $\mathbb{K}$ parallel to $\mathbb{E}$ and $\mathbb{X}$. The linear function $f : \mathbb{K} \longrightarrow \mathbb{K}$ is the restriction of the action of $\mathbf{M}_\zeta$ onto the contractive space $\mathbb{K}$, that is, for all $\mathbf{x} \in \mathbb{R}^m$ we have $f \circ \pi(\mathbf{x}) = \pi(\mathbf{M}_\zeta \mathbf{x})$. By construction we clearly have that f is a contraction.

The Rauzy fractal $\mathfrak{R}_\zeta$ associated to ζ is defined by

$$\mathfrak{R}_\zeta := \overline{\{\pi(\mathbf{l}(u_0 \cdots u_n)) : n \in \mathbb{N}\}} \subset \mathbb{K}, \tag{2}$$

where $\mathbf{u} = u_0 u_1 u_2 \cdots$ is a periodic point of ζ. The Rauzy fractal naturally decomposes into $|\mathcal{A}|$ subsets. In particular, for each $a \in \mathcal{A}$ define

$$\mathfrak{R}_\zeta(a) := \overline{\{\pi(\mathbf{l}(u_0 \cdots u_{n-1})) : n \in \mathbb{N}, u_n = a\}} \subset \mathfrak{R}_\zeta \tag{3}$$

and we obviously have

$$\mathfrak{R}_\zeta = \bigcup_{a \in \mathcal{A}} \mathfrak{R}_\zeta(a). \tag{4}$$

The collection $\{\mathfrak{R}_\zeta(a) \,|\, a \in \mathcal{A}\}$ is called the *natural partition* of the Rauzy fractal $\mathfrak{R}_\zeta$. Each $\mathfrak{R}_\zeta(a)$ and, hence, the Rauzy fractal $\mathfrak{R}_\zeta$ itself, is the closure of its interior and has positive Lebesgue measure (*cf.* [25]).

The collection $\{\mathfrak{R}_\zeta(a) : a \in \mathcal{A}\}$ is the unique non-empty solution of the graph directed iterated function system (GIFS for short) in the sense of [15] that satisfies the system of set equations

$$\mathfrak{R}_\zeta(a) = \bigcup_{\zeta(b) = PaS} (\pi(\mathbf{l}(P)) + f(\mathfrak{R}_\zeta(b))) \qquad (a \in \mathcal{A}). \tag{5}$$

This union is measure-theoretically disjoint (*cf.* [12, 25]). In contrast, the union (4) is measure-theoretically disjoint provided that ζ satisfies the *strong coincidence condition* (*cf.* [5, 25]) which means that we either have

Positive coincidence: for all $a_1, a_2 \in \mathcal{A}$ there exist $k \in \mathbb{N}$, $b \in \mathcal{A}$, $P_1, S_1, P_2, S_2 \in \mathcal{A}^*$ such that $\zeta^k(a_1) = P_1 b S_2$, $\zeta^k(a_2) = P_2 b S_2$ and $\mathbf{l}(P_1) = \mathbf{l}(P_2)$ or

Negative coincidence: for all $a_1, a_2 \in \mathcal{A}$ there exist $k \in \mathbb{N}$, $b \in \mathcal{A}$, $P_1, S_1, P_2, S_2 \in \mathcal{A}^*$ such that $\zeta^k(a_1) = P_1 b S_2$, $\zeta^k(a_2) = P_2 b S_2$ and $\mathbf{l}(S_1) = \mathbf{l}(S_2)$.

It is conjectured that if ζ is irreducible then the coincidence condition is satisfied (coincidence conjecture). Up to now this is only proved for the 2-letter case (see [7]).

A substitution ζ may have more than one periodic point, however, due to primitivity the Rauzy fractal does not depend on the actual choice of $\mathbf{u}$. We immediately see that a higher power of ζ induces the same Rauzy fractal, i.e., for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$ we have $\mathfrak{R}_\zeta(a) = \mathfrak{R}_{\zeta^k}(a)$. Therefore, without loss of generality, we may assume $\mathbf{u}$ to be a fixed point of ζ. Note that if ζ has only one periodic point (which is necessarily a fixed point) then the strong coincidence condition is automatically satisfied (*cf.* [25]).

Suppose that ζ satisfies the strong coincidence condition. Define the lattice Γ by

$$\Gamma := \{\pi(x_1, \dots, x_m) : (x_1, \dots, x_m) \in \mathbb{Z}^m, x_1 + \cdots + x_m = 0\} \subset \mathbb{K}.$$

We say that the Rauzy fractal $\mathfrak{R}_\zeta$ has the *tiling property* if it induces a proper lattice tiling with respect to Γ, that is, $\bigcup_{\gamma\in\Gamma}(\gamma+\mathfrak{R}_\zeta)=\mathbb{K}$, and for distinct elements $\gamma_1,\gamma_2\in\Gamma$ we always have $\mathrm{int}(\gamma_1+\mathfrak{R}_\zeta)\cap\mathrm{int}(\gamma_2+\mathfrak{R}_\zeta)=\emptyset$. A priori, only substitutions that satisfy the so-called *quotient mapping condition* come into question (see [20, Definition 3.13]). Observe that this condition is fulfilled by any irreducible substitution. In this case the collection $\{\gamma+\mathfrak{R}_\zeta : \gamma\in\Gamma\}$ provides a multiple tiling of $\mathbb{K}$. The step towards a proper tiling is tricky. It is conjectured that each irreducible unit Pisot substitution induces a proper tiling (Pisot conjecture, see, for example, [4]). Besides this there exist several tiling conditions; here we refer the interested reader to [10, 20] and the references therein.

Throughout the entire article we suppose that all substitutions satisfy the strong coincidence condition and have the tiling property without mentioning this any more. One sufficient condition that implies that the multi-tiling is a proper tiling is that the origin is an inner point of the Rauzy fractal $\mathfrak{R}_\zeta$. We follow the notation from [20] and call this characteristic the *geometric property (F)*. Usually it is quite tricky to decide whether ζ has this property. More precisely, the verification runs algorithmically via graphs (see, for example, [20, Theorem 4.6]). Following [9] (see also [3]) the geometric property (F) is equivalent to the fact that the Dumont-Thomas numeration induced by ζ has the (algebraic) extended finiteness property which is a generalization of the finiteness property for beta-expansions introduced in [13].

2.3. **The balanced pair algorithm.** Let U and V be two finite words. We said that (U,V) is a *balanced pair* if the occurrences of each letter of the alphabet in U and V are the same, i.e., $\mathbf{l}(U)=\mathbf{l}(V)$. This clearly implies that U and V are of the same length. If for all proper prefixes U' of U and V' of V the pair (U',V') is not balanced then we say that the balanced pair (U,V) is *minimal*.

Let ζ_1 and ζ_2 be two primitive substitutions having the same incidence matrix. Then for each balanced pair (U,V) the pair $(\zeta_1(U),\zeta_2(V))$ is again balanced.

We may assume that ζ_1 and ζ_2 possess the one-sided fixed points $\mathbf{u}_1$ and $\mathbf{u}_2$, possibly by considering higher powers (which does not change the Rauzy fractals). An *initial balanced pair* for $(\mathbf{u}_1,\mathbf{u}_2)$ is a balanced pair (U,V) such that U and V are proper prefixes of $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. Clearly, if (U,V) is an initial balanced pair then $(\zeta_1(U),\zeta_2(V))$ is also an initial balanced pair.

If we successively apply ζ_1 and ζ_2 on an initial balanced pair and subsequently split the result into minimal balanced pairs we obtain a representation of $\mathbf{u}_1$ and $\mathbf{u}_2$ as

$$\mathbf{u}_1=U_1U_2U_3\cdots, \quad \mathbf{u}_2=V_1V_2V_3\cdots$$

such that (U_n,V_n) is a minimal balanced pair for all $n\in\mathbb{N}$. Let

$$\mathcal{B}:=\{(U_n,V_n): n\in\mathbb{N}\}$$

be the set of all occurring minimal balanced pairs.

Definition 1 (Balanced pair algorithm)**.** Let ζ_1 and ζ_2 be two primitive substitutions over the same alphabet and with the same incidence matrix. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are fixed points of ζ_1 and ζ_2, respectively.

- We say that the balanced pair algorithm is defined if there exists an initial balanced pair for $(\mathbf{u}_1,\mathbf{u}_2)$.

- We say that the balanced pair algorithm terminates if the set $\mathcal{B}$ of minimal balanced pairs is finite.

In [18] we find the following result:

Proposition 1 ([18, Lemma 4.3])**.** *Let ζ_1 and ζ_2 be unimodular irreducible Pisot substitutions with the same incidence matrix, whose Rauzy fractals are $\mathfrak{R}_{\zeta_1}$, $\mathfrak{R}_{\zeta_2}$, respectively. Suppose that ζ_1 has the geometric property (F). Then the balanced pair algorithm is defined and terminates.*

We generalise this result by weakening the hypothesis from [18]. In particular, we also consider reducible substitutions as long as they induce a proper lattice tiling (tiling property). Furthermore, we do not require the geometric property (F). In this setting we also obtain as necessary result that if the intersection of the Rauzy fractals $\mathfrak{R}_{\zeta_1}$ and $\mathfrak{R}_{\zeta_2}$ has empty interior then the balanced pair algorithm is not defined or does not terminate.

Theorem 1. *Let ζ_1 and ζ_2 be two primitive, unimodular Pisot substitutions with the same incidence matrix satisfying the strong coincidence condition and having the tiling property. The balanced pair algorithm of ζ_1 and ζ_2 is defined and terminates if and only if $\mathrm{int}(\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}) \neq \emptyset$.*

Concerning the general structure of $\mathfrak{R}_1 \cap \mathfrak{R}_2$ we have two "good" cases, namely, when $\mathfrak{R}_1 \cap \mathfrak{R}_2$ has non-empty interior and when $\mathfrak{R}_1 \cap \mathfrak{R}_2$ consists only of the origin. The first case is covered by Theorem 1. In the second case it is quite easy to see that the balanced pair algorithm is not defined. Theoretically the intersection may also have empty interior but contain points different from the origin. Here we have no information whether the balanced pair algorithm is defined or not.

3. Proof of Theorem 1

Throughout the section we let ζ_1 and ζ_2 be unimodular Pisot substitutions with the same incidence matrix satisfying the strong coincidence condition and having the tiling property, and $\mathfrak{R}_{\zeta_1}$, $\mathfrak{R}_{\zeta_2}$ be their associated Rauzy fractals. Without loss of generality we may assume that the substitutions have the one-sided fixed points $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively.

Lemma 1. *If $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ has non-empty interior, then the following items hold.*

(a) *The origin is not an isolated point of $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$.*
(b) *If $U \in \mathcal{A}^*$ is a proper prefix of $\mathbf{u}_1$ and $\pi(\mathbf{l}(U)) \in \mathrm{int}(\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2})$, then (U, V) is a balanced pair where V is the (uniquely determined) proper prefix of $\mathbf{u}_2$ whose length is equal to the length of U.*

Proof. Let $X := \mathrm{int}(\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2})$. To show (a) observe that f is a contraction with the origin as a fixed point, and by the definition of the Rauzy fractals we have $f(R_{\zeta_1}) \subset R_{\zeta_1}$ and $f(R_{\zeta_2}) \subset R_{\zeta_2}$. Thus, for all $n \geq 1$ we have $f^n(X) \subset X$ which implies that X contains a sequence that converges to the origin.

In order to prove (b) we let $U \in \mathcal{A}^*$ be a prefix of $\mathbf{u}_1$ such that $\pi(\mathbf{l}(U)) \in X$. Let $V \in \mathcal{A}^*$ be the prefix of $\mathbf{u}_2$ of the same length, that is $|U| = |V|$. We show by contradiction that (U, V) is a balanced pair. Hence, suppose that $\mathbf{l}(U) \neq \mathbf{l}(V)$. Set $\mathbf{x} := \mathbf{l}(U) - \mathbf{l}(V)$ and observe that $\gamma := \pi(\mathbf{x}) \in \Gamma$. By definition, $\pi(\mathbf{l}(V)) \in \mathfrak{R}_{\zeta_2}$, hence, $\gamma + \pi(\mathbf{l}(V))$ is contained in the translate $\gamma + \mathfrak{R}_{\zeta_2}$. On the other hand, by construction $\gamma + \pi(\mathbf{l}(V)) = \pi(\mathbf{l}(U)) \in X \subset \mathrm{int}(\mathfrak{R}_{\zeta_2})$. This clearly contradicts the tiling property. $\square$

We obtain the first part of Theorem 1.

Lemma 2. *If $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ has non empty interior then the balanced pair algorithm of ζ_1 and ζ_2 is defined and terminates.*

Proof. The balanced pair algorithm terminates if (and only if) the fixed words $\mathbf{u}_1$ and $\mathbf{u}_2$ can be decomposed as

$$\mathbf{u}_1 = U_1U_2U_3U_4\cdots, \qquad \mathbf{u}_2 = V_1V_2V_3V_4\cdots$$

such that the collection $\{(U_k, V_k) : k \in \mathbb{N}\}$ is a finite set of minimal balanced pairs.

We show that $\mathbf{u}_1$ and $\mathbf{u}_2$ can be decomposed into non-empty words

$$\mathbf{u}_1 = U_1'U_2'U_3'U_4'\cdots, \qquad \mathbf{u}_2 = V_1'V_2'V_3'V_4'\cdots$$

of bounded length such that (U_n', V_n') is a (not necessarily minimal) balanced pair for each $n \in \mathbb{N}$. Due to the boundedness of the words this is equivalent to the previous condition.

As $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ has non-empty interior there exists an $a \in \mathcal{A}$ such that $\mathfrak{R}_{\zeta_1}(a) \cap \mathfrak{R}_{\zeta_2}$ has non-empty interior. Now consider the respective equation (5). We can iterate it k times and obtain

$$\mathfrak{R}_{\zeta_1}(a) = \bigcup_{\zeta_1^k(b)=PaS} \Big(\pi(\mathbf{l}(P)) + f^k(\mathfrak{R}_{\zeta_1}(b))\Big).$$

Since f is a contraction we can choose k sufficiently large such that one of the subsets of this union is completely contained in the interior of $\mathfrak{R}_{\zeta_1}(a) \cap \mathfrak{R}_{\zeta_2}$. In other words, there exist particular $k \in \mathbb{N}$, $b \in \mathcal{A}$, $P, S \in \mathcal{A}^*$ such that $\zeta_1^k(b) = PaS$ and

$$\pi(\mathbf{l}(P)) + f^k(\mathfrak{R}_{\zeta_1}(b)) \subset \operatorname{int}(\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}).$$

Now subdivide $\mathbf{u}_1$ according to the occurrences of b. In particular, for each $n \in \mathbb{N}$ let $X_n \in \mathcal{A}^*$ such that $|X_n|_b = 0$ and $\mathbf{u}_1 = X_1bX_2bX_3b\cdots$. Set $U_1' := \zeta_1^k(X_1)P$ and $U_n' := aS\zeta_1^k(X_n)P$ for $n \geq 2$. Since b occurs in $\mathbf{u}_1$ with bounded gaps the collection $\{X_n : n \in \mathbb{N}\}$ is a finite set and, hence, $|U_n'| \leq K$ for all $n \in \mathbb{N}$ with $K = \max\{|\zeta_1^k(X_nb)| : n \in \mathbb{N}\}$.

Since $\mathbf{u}_1$ is a fixed point of ζ_1 we have

$$\mathbf{u}_1 = \zeta_1^k(X_1)\zeta_1^k(b)\zeta_1^k(X_2)\zeta_1^k(b)\zeta_1^k(X_3)\zeta_1^k(b)\cdots = U_1'U_2'U_3'\cdots.$$

For each $N \in \mathbb{N}$ let V_n' be the word of length $|U_n'|$ such that $\mathbf{u}_2 = V_1'V_2'V_3'\cdots$.

By (3) we have

$$\mathfrak{R}_{\zeta_1}(b) = \overline{\{\pi(\mathbf{l}(X_1bX_2b\cdots bX_n)) : n \in \mathbb{N}\}}.$$

This immediately shows that

$$\begin{aligned} f^k(\mathfrak{R}_{\zeta_1}(b)) + \pi(\mathbf{l}(P)) &= \overline{\{\pi(\mathbf{l}(\zeta_1^k(X_1bX_2b\cdots bX_n)P)) : n \in \mathbb{N}\}} \\ &= \overline{\{\pi(\mathbf{l}(U_1'U_2'\cdots U_n')) : n \in \mathbb{N}\}}. \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$ we have that $\pi(\mathbf{l}(U_1'U_2'\cdots U_n'))$ is contained in the interior of $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$. By Lemma 1 we conclude that $(U_1'U_2'\cdots U_n', V_1'V_2'\cdots V_n')$ is an initial balanced pair and, hence, (U_j', V_j') is a balanced pair of length at most K for all $j \geq 1$. □

Lemma 3. *If the balanced pair algorithm of ζ_1 and ζ_2 is defined and terminates then $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ has non-empty interior.*

Proof. Suppose that the balanced pair algorithm terminates, that is there exists a sequence of minimal balanced pairs $(U_n, V_n)_{n\geq 1}$ such that $\mathbf{u}_1 = U_1U_2U_3\cdots$ and $\mathbf{u}_2 = V_1V_2V_3\cdots$. Furthermore, for each $n \geq 1$ we have

$$\pi(\mathbf{l}(U_1U_2\cdots U_n)) = \pi(\mathbf{l}(V_1V_2\cdots V_n)) \in \mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}.$$

Since Rauzy fractals are compact sets we have that

$$\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2} \supseteq \overline{\{\pi(\mathbf{l}(U_1U_2\cdots U_n)) : n \geq 1\}} = \overline{\{\pi(\mathbf{l}(V_1V_2\cdots V_n)) : n \geq 1\}}.$$

The set $\mathcal{B}$ of minimal balanced pairs is finite and we can associate with each $B = (U, V) \in \mathcal{B}$ the subset

$$\mathfrak{R}(B) := \overline{\{\pi(\mathbf{l}(U_1U_2\cdots U_n)) : n \geq 1, U_{n+1} = U\}} = \overline{\{\pi(\mathbf{l}(V_1V_2\cdots V_n)) : n \geq 1, V_{n+1} = V\}}$$

and, clearly,

$$\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2} \supseteq \bigcup_{B\in\mathcal{B}} \mathfrak{R}(B).$$

For convenience for each $B = (U, V) \in \mathcal{B}$ we let $\mathbf{l}(B) := \mathbf{l}(U)(= \mathbf{l}(V))$. Define a substitution ζ_{12} over $\mathcal{B}$ by

$$\zeta_{12} : \mathcal{B} \longrightarrow \mathcal{B}^*, (U, V) \longmapsto (U_1', V_1')(U_2', V_2')\cdots(U_\ell', V_\ell')$$

whenever $\zeta_1(U) = U_1'U_2'\cdots U_\ell'$ and $\zeta_2(V) = V_1'V_2'\cdots V_\ell'$. We see that the collection $\{\mathfrak{R}(B) : B \in \mathcal{B}\}$ satisfies a GIFS since for each $B \in \mathcal{B}$ we have

$$\mathfrak{R}(B) = \bigcup_{\zeta_{12}(B')=B_1\cdots B_kBB_1'\cdots B_{k'}'} \big(\pi(\mathbf{l}(B_1\cdots B_k)) + f(\mathfrak{R}_\zeta(B'))\big) \tag{6}$$

(*cf.* [18, Theorem 4.4]). Note that $\mathfrak{R}(B)$ is a subset of $\mathfrak{R}_{\zeta_1}(u_1)$ and $\mathfrak{R}_{\zeta_2}(v_1)$, respectively, where u_1 is the first letter of U, v_1 is the first letter of V, and $B = (U, V)$. Therefore, the disjointness of the union (5) for ζ_1 (and also ζ_2) immediately implies the disjointness of (6).

By [18, Lemma 4.5] the corresponding subdivision matrix (which is the transpose of the incidence matrix $\mathbf{M}_{\zeta_{12}}$) possesses a dominant root that coincides with that of $\mathbf{M}_{\zeta_1}$ and $\mathbf{M}_{\zeta_2}$. Observe that it is not clear whether the underlying graph is strongly connected. But it must possess a strongly connected component whose incidence matrix has β as a dominant root. Therefore, from [15, Theorem 4] it follows that the subsets $\mathfrak{R}(B)$ that correspond to the vertices of this strongly connected components have positive d-dimensional Lebesgue measure and, hence, non-empty interior which implies that $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ has non-empty interior. □

Now, Theorem 1 follows directly from Lemma 2 and Lemma 3.

Observe that $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ is not necessarily the closure of its interior (even if this interior is non-empty). Therefore, the substitution ζ_{12} only yields information about the closure of the interior of $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$. Several examples in [18, 19] show that the substitution ζ_{12} is usually not irreducible. In fact, it is not even clear whether it is always primitive. For this reason we do not say that (the closure of the interior of) the intersection $\mathfrak{R}_{\zeta_1} \cap \mathfrak{R}_{\zeta_2}$ is given by a Rauzy fractal.

Problem 1. If the balanced pair algorithm terminates, what is the shape of ζ_{12}? Is this substitution always primitive? What can be said about the additional eigenvalues?

A second question concerns the intersection $\mathfrak{R}_1 \cap \mathfrak{R}_2$. We already discussed the possible structures of $\mathfrak{R}_1 \cap \mathfrak{R}_2$ in the introduction and we now see that there exist two "pathological" cases. On one hand the intersection may contains points different from the origin but have empty interior. On the other hand the intersection may have non-empty interior without being the closure of its interior. However, we do not know whether these cases really occur.

Problem 2. Characterise the structure of $\mathfrak{R}_1 \cap \mathfrak{R}_2$. Which cases can occur and what can be said about the balanced pair algorithm for these cases?

Remark 1. One easily generalises the result of Theorem 1 in order to analyse the intersection $\mathfrak{R}_1 \cap \cdots \cap \mathfrak{R}_n$ of Rauzy fractals induced by substitutions $\zeta_1, \ldots, \zeta_n$ with n larger than 2. For this purpose we have to define (minimal) balanced tuples and the balanced tuples algorithm in a straightforward way. The respective proofs run analogously and are left to the interested reader.

4. The geometric property (F)

In the present section we show that our results provide a necessary combinatorial condition for a unimodular Pisot substitution ζ to have the geometric property (F). For a word $V = v_0 \ldots v_n \in \mathcal{A}^*$ we denote its reversed word by $\overline{V} := v_n \ldots v_0$. The *reversed substitution* $\overline{\zeta}$ of ζ is defined by $\overline{\zeta}(a) := \overline{\zeta(a)}$, for all $a \in \mathcal{A}$. Clearly ζ and $\overline{\zeta}$ have the same incidence matrix and $\mathbf{v} = v_1 v_2 v_3 \cdots \in \mathcal{A}^{\mathbb{N}}$ is a periodic point of $\overline{\zeta}$ if and only if $\overline{\mathbf{v}} = \cdots v_3 v_2 v_1$ is a left periodic point with respect to ζ, that is

$$\zeta(\overline{\mathbf{v}}) = \cdots \zeta(v_3)\zeta(v_2)\zeta(v_1) = \cdots v_3 v_2 v_1 = \overline{\mathbf{v}}.$$

Proposition 2. *Let ζ be a unimodular Pisot substitution that has the geometric property (F). Then* $\mathrm{int}(\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}) \neq \emptyset$.

Proof. In [19] is was shown that $\mathfrak{R}_{\overline{\zeta}} = -\mathfrak{R}_\zeta$. Hence, if $\mathbf{0} \in \mathrm{int}(\mathfrak{R}_\zeta)$ then $\mathbf{0} \in \mathrm{int}(\mathfrak{R}_{\overline{\zeta}})$ and therefore $\mathrm{int}(\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}) \neq \emptyset$. □

This immediately yields the following condition.

Corollary 1. *Let ζ be a unimodular Pisot substitution that has the geometric property (F). Then for the pair $(\mathbf{u}, \mathbf{v})$ of infinite words such that $\mathbf{u}$ is a (right) periodic point of ζ and $\overline{\mathbf{v}}$ is a left periodic point of ζ the balanced pair algorithm is defined and terminates.*

We conclude the article with a necessary condition on a unimodular Pisot substitution ζ that ensures that $\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}$ has empty interior, namely, in the case where at least one of the contractive eigenvalues of the incidence matrix is real and positive. The interesting thing about this criterion is that it only relies on the spectral properties of the matrix $\mathbf{M}_\zeta$ and not on the balanced pair algorithm.

Theorem 2. *Let ζ be a unimodular Pisot substitution over the alphabet $\mathcal{A} = \{1, \ldots, m\}$. Suppose that there exists a contractive eigenvalue α that is real and positive and the scalar products $\mathbf{v}_\alpha \cdot \mathbf{l}(P)$ of the proper prefixes P of $\zeta(j)$ for each $j \in \mathcal{A}$ are all non-negative or non-positive, where $\mathbf{v}_\alpha$ is a left eigenvector of $\mathbf{M}_\zeta$ with respect to α. Then $\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}$ has empty interior.*

Proof. We organise the d Galois conjugates different from β as follows: $\alpha_1, \alpha_2, \ldots, \alpha_d$ such that $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$. $\alpha_1 = \alpha$, and $\alpha_{t+1}, \ldots, \alpha_{t+2s} \in \mathbb{C} \setminus \mathbb{R}$ (hence, $t + 2s = d$). Without loss of generality we may suppose that $\alpha_{t+k} = \overline{\alpha_{t+s+k}}$ for each $k \in \{1, \ldots, s\}$ (where $\overline{\alpha_{t+s+k}}$ denotes the complex conjugate of α_{t+k}).

For each $k \in \{1, \dots, d\}$ denote by $\mathbf{v}_{\alpha_k}$ a left eigenvalue of $\mathbf{M}_\zeta$ with respect to α_k (hence, $\mathbf{v}_{\alpha_1} = \mathbf{v}_\alpha$). We define the matrix $\mathbf{P}$ to be the real $d \times m$ matrix whose first t rows are the vectors $\mathbf{v}_{\alpha_k}$ for $k \in \{1, \dots, t\}$ while the $(t+2k-1)$st and $(t+2k)$th row are given by $\Re(\mathbf{v}_{\alpha_{t+k}})$ and $\Im(\mathbf{v}_{\alpha_{t+k}})$, respectively, for each $k \in \{1, \dots, s\}$. Observe that the multiplication with $\mathbf{P}$ is a suitable representation of the projection π in some basis. Now, the map f is given by the diagonal block matrix

$$\mathbf{D} := \operatorname{diag}\left(\alpha_1, \dots, \alpha_t, \begin{pmatrix} \Re(\alpha_{t+1}) & -\Im(\alpha_{t+1}) \\ \Im(\alpha_{t+1}) & \Re(\alpha_{t+1}) \end{pmatrix}, \dots, \begin{pmatrix} \Re(\alpha_{t+s}) & -\Im(\alpha_{t+s}) \\ \Im(\alpha_{t+s}) & \Re(\alpha_{t+s}) \end{pmatrix}\right).$$

In other words, for each $\mathbf{x} \in \mathbb{R}^d$ we have

$$\mathbf{P} \cdot \mathbf{M}_\zeta \cdot \mathbf{x} = \mathbf{D} \cdot \mathbf{P} \cdot \mathbf{x}.$$

We have

$$\mathfrak{R}_\zeta := \overline{\{\mathbf{P} \cdot \mathbf{l}(u_0 \cdots u_n) : n \in \mathbb{N}\}}.$$

It is well known that each prefix $u_1 \cdots u_n$ of the fixed point $\mathbf{u}$ can be represented uniquely as

$$u_1 \cdots u_n = \zeta^k(P_k)\zeta_{k-1}(P_{k-1}) \cdots \zeta(P_1)P_0$$

where, for each $r \in \{0, \dots k\}$, P_r is a proper prefix of $\zeta(j)$ for some $j \in \mathcal{A}$; for details see, for example, [9, 11]. Since $\mathbf{M}_\zeta \cdot \mathbf{l}(A) = \mathbf{l}(\zeta(A))$, we thus have

$$\mathbf{P} \cdot \mathbf{l}(u_1 \cdots u_n) = \sum_{r=0}^{k} \mathbf{P} \cdot \mathbf{M}_\zeta^r \cdot \mathbf{l}(P_r) = \sum_{r=0}^{k} \mathbf{D}^r \cdot \mathbf{P} \cdot \mathbf{l}(P_r). \tag{7}$$

Now observe that by construction the first entry of $\mathbf{P} \cdot \mathbf{l}(P_r)$ is given by the scalar product $\mathbf{v}_\alpha \cdot \mathbf{l}(P_r)$. By the assumption of the theorem all these products are non-negative or non-positive where we may assume, without loss of generality, that $\mathbf{v}_\alpha \cdot \mathbf{l}(P_r) \geq 0$ for all $r \in \{0, \dots, k\}$. By the positivity of α and the structure of $\mathbf{D}$ we therefore have that the first entry of $\mathbf{P} \cdot \mathbf{l}(u_1 \cdots u_n)$ is non-negative for all $n \in \mathbb{N}$ and, hence,

$$\mathfrak{R}_\zeta \subset \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \geq 0\}.$$

On the other hand,

$$\mathfrak{R}_{\overline{\zeta}} = -\mathfrak{R}_\zeta \subset \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \leq 0\}.$$

Therefore, $\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}$ has empty interior. □

If the condition on the prefixes are strict inequalities then we even obtain a stronger result.

Corollary 2. *Let ζ be a unimodular Pisot substitution over the alphabet $\mathcal{A} = \{1, \dots, m\}$. Suppose that there exists a contractive eigenvalue α that is real and positive and the scalar products $\mathbf{v}_\alpha \cdot \mathbf{l}(P)$ of the proper prefixes P of $\zeta(j)$ for each $j \in \mathcal{A}$ are all strictly negative or positive, where $\mathbf{v}_\alpha$ is a left eigenvector of $\mathbf{M}_\zeta$ with respect to α. Then $\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}} = \{\mathbf{0}\}$.*

Proof. Without loss of generality we may assume that the scalar products are strictly positive. We consider the proof of Theorem 2 under the modified conditions and see that several inequalities become strict. In particular, if $\mathbf{u} = u_1u_2u_3 \cdots$ then

$$Y := \{\mathbf{P} \cdot \mathbf{l}(u_1u_2 \cdots u_n) : n \in \mathbb{N}\} \subset \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \geq 0\}.$$

And from (7) we see that if the limit point of a convergent sequence of points in Y is not contained in this half then it must be the origin. □

We see that algebraic properties (a real eigenvalue and the condition on the prefixes) is reflected in a combinatorial property on the language (there is either no initial balanced pair or the balanced pair algorithm does not terminate). Note that this result somehow generalises [1, Proposition 1] which says if an algebraic integer β has the algebraic finiteness property (F) then β is the unique positive real Galois conjugate. Especially, if ζ is a beta-substitution and the dominant root β possesses a real positive algebraic conjugate (different from β) then we immediately obtain that $\mathrm{int}(\mathfrak{R}_\zeta \cap \mathfrak{R}_{\overline{\zeta}}) = \emptyset$ since beta-substitutions always satisfy the condition on the prefixes.

Example 1. Consider the family of substitutions

$$\zeta_a : 1 \to 1^a 23, \quad 2 \to 1, \quad 3 \to 13.$$

with $a \geq 1$. The characteristic polynomial of the incidence matrix is $x^3-(1+a)x^2+(a-2)x+1$. Using the intermediate value theorem it can easily be proved that it always has three real roots: a dominant one $\beta > 1$, a positive one $\alpha_1 \in (0,1)$ and a negative one $\alpha_2 \in (-1,0)$. A left eigenvector with respect to α_1 is given by $\mathbf{v}_{\alpha_1} = (1-\alpha_1, (1-\alpha_1)\alpha_1^{-1}, -1)$. The substitution ζ_a induces two types of proper prefixes. On one hand $P = 1^k$ for $1 \leq k \leq a$ which yields $\mathbf{v}_\alpha \cdot \mathbf{l}(P) = k(1-\alpha_1) > 0$. On the other hand, we have $P = 1^a 2$ which yields $\mathbf{v}_\alpha \cdot \mathbf{l}(P) = (a+\alpha^{-1})(1-\alpha_1) > 0$. This shows that the conditions of Corollary 2 are satisfied and, hence, the intersection of the Rauzy fractals $\mathfrak{R}_{\zeta_a}$ and $\mathfrak{R}_{\overline{\zeta_a}}$ consists of the origin only (see Figure 1). This immediately implies that that the balanced pair algorithm is not defined for the one-sided fixed points of ζ_a and the reversed substitution $\overline{\zeta_a}$. For the particular case ζ_1 the absence of an initial balanced pair has been conjectured in [19, Example 5] and finally proved in [24] using very different (combinatorial) techniques.

Observe that the condition on the prefixes is necessary and cannot be omitted. Indeed, lets consider the substitution ζ_a' obtained from ζ_a by changing the order ("flipping") of the letters of the image of 3, that is

$$\zeta_a' : 1 \to 1^a 23, \quad 2 \to 1, \quad 3 \to 31.$$

The incidence matrix of ζ_a' coincides with that of ζ_a but the condition on the prefixes does not hold anymore since for the prefix $P = 3$ we have $\mathbf{v}_\alpha \cdot \mathbf{l}(P) = -1 < 0$. The Rauzy fractal for the case $a = 1$ is shown on the right hand side of Figure 1.

Example 2. Here we consider substitutions of the form

$$\zeta : 1 \to 1^{a-1}2, \quad 2 \to 1^{a-b-1}3, \quad 3 \to 1^{a-b}3,$$

with a, b integers such that $2 \leq b \leq a-1$. The Perron-Frobenius eigenvalue β of $\mathbf{M}_\zeta$ is the dominant root of $x^3 - ax^2 + bx - 1$. More precisely, the substitution ζ is the beta-substitution associated with the beta-expansion with respect to (the non-simple Parry number) β (see [2, 8]). For details on beta-substitutions we refer to the survey article [9].

Observe that for any choice of a and b the substitution ζ does not have the geometric property (F) which is (for beta-substitutions) equivalent to the fact that the corresponding beta-expansions do not satisfy the finiteness property (F) introduced in [13]. It depends on a and b whether the incidence matrix possess (non-dominant) real roots or a pair of complex conjugate roots. For example, in the case $a = 3$ and $b = 2$ we have a pair of complex roots. Here the balanced pair algorithm between the one-sided fixed points of this substitution and its reversed substitution is defined and it terminates. The Rauzy fractal intersects with the

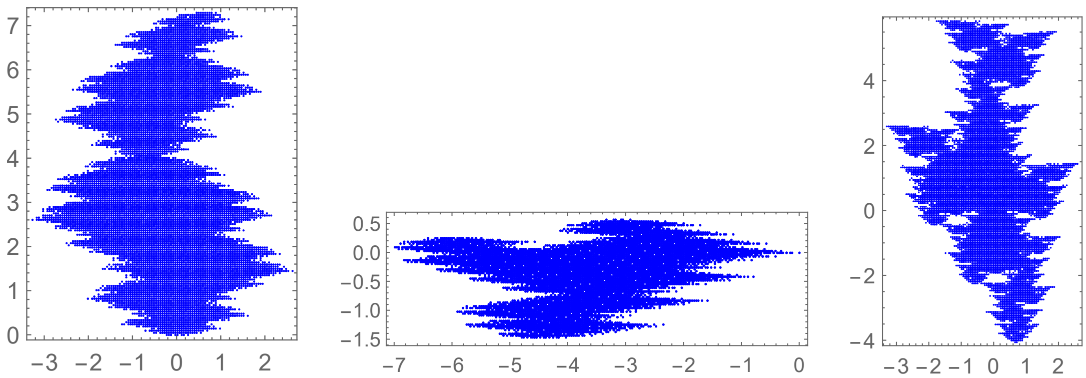

FIGURE 1. Example 1: on the left we see $\mathfrak{R}_{\zeta_1}$ while the centre shows $\mathfrak{R}_{\zeta_3}$. Both are contained in the upper half plain - the intersection with the Rauzy fractals associated with the reversed substitution has always empty interior. On the right the Rauzy fractal induced by the flipped substitution ζ_1' is depicted. We clearly see that here $\operatorname{int}(\mathfrak{R}_{\zeta_1'} \cap \mathfrak{R}_{\overline{\zeta_1'}}) \neq \emptyset$.

reversed one (with positive measure - see Figure 2). This shows that equivalence does not hold in Proposition 2 (and hence, Corollary 1).

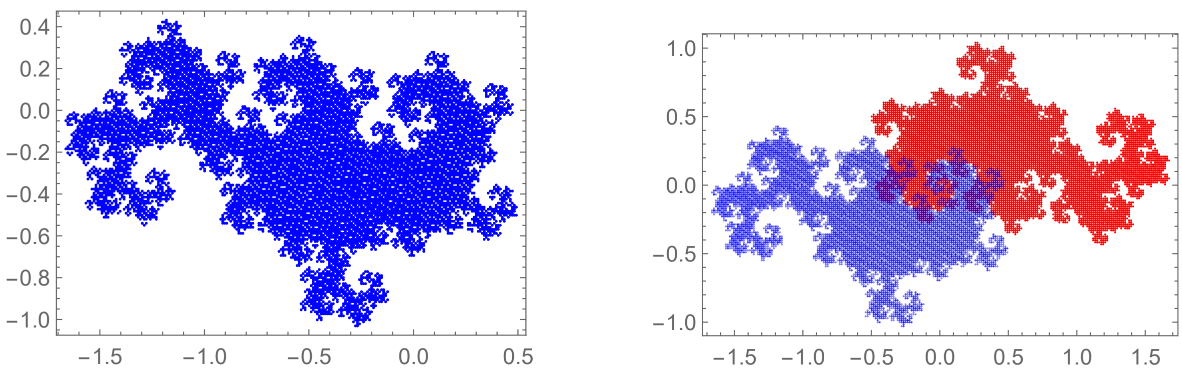

FIGURE 2. The Rauzy fractal $\mathfrak{R}_\zeta$ of Example 2 for $a = 3$ and $b = 2$ (left). On the right we see the intersection of $\mathfrak{R}_\zeta$ with the Rauzy fractal $\mathfrak{R}_{\overline{\zeta}}$ induced by the reversed substitution $\overline{\zeta}$.

If we choose $a = 6$ and $b = 5$ then the non-dominant roots are real[1] and positive, hence the conditions of Proposition 2 are satisfied. The intersection of the Rauzy fractals $\mathfrak{R}_\zeta$ and $\mathfrak{R}_{\overline{\zeta}}$ has empty interior (see the left hand side in Figure 3).

Observe that it does not depend on the conjugates of the dominant root β (real or complex) whether the Rauzy fractal $\mathfrak{R}_\zeta$ properly intersects with $\mathfrak{R}_{\overline{\zeta}}$. Indeed, for $a = 5$ and $b = 4$ the matrix $\mathbf{M}_\zeta$ has complex conjugate roots but the Rauzy fractals do not seem to intersect (see the right hand side in Figure 3).

[1]This seems to be the real setting with the smallest coefficients; the next "real" one is given by $a = 7$ and $b = 6$.

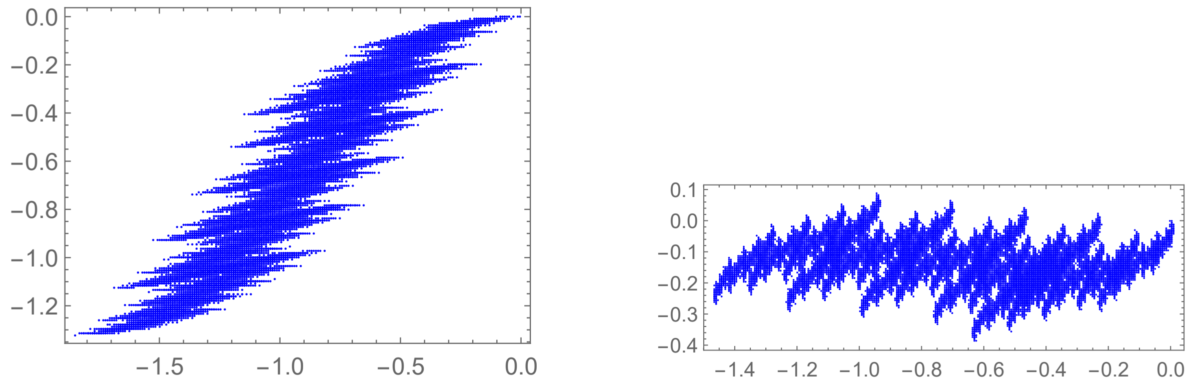

FIGURE 3. The Rauzy fractals of Example 2 for $a = 6$, $b = 5$ (left) and $a = 5$, $b = 4$ (right).

Problem 3. Give conditions on a and b such that for this class of substitutions the intersection of $\mathfrak{R}_\zeta$ and $\mathfrak{R}_{\overline{\zeta}}$ has non-empty interior.

Example 3. We consider substitutions of the type

$$1 \to 1^a 2, \quad 2 \to 1^{b-1} 3, \quad 3 \to 1^{a-1} 2;$$

for $1 \leq b < a$. These substitutions are beta-substitutions with respect to the dominant root of $x^3 - ax^2 - bx + 1$ (*cf.* [8, Theorem 2]). Again, they do not satisfy the geometric property (F). Using the intermediate value theorem one can easily check that for any choice of a and b the polynomial has exactly one positive and one negative (non-dominant) real root. The substitutions satisfy the conditions of Theorem 2, therefore the intersections of the Rauzy fractals has empty interior.

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