SUBSTITUTIVE NUMBER SYSTEMS

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ABSTRACT. In the present article we associate a primitive substitution with a family of non-integer positional number systems with respect to the same base but with different sets of digits. In this way we generalise the classical Dumont-Thomas numeration which corresponds to one specific case. Therefore, our concept also covers beta-expansions induced by Parry numbers. But we establish links to variants of beta-expansions such as symmetric beta-expansions, too. In other words, we unify several well-known notions of non-integer representations within one general framework. A focus in our research is set on finiteness and periodicity properties. It turns out that these characteristics mainly depend on the substitution. As a consequence we are able to relate known finiteness properties that are viewed independently yet.

1. INTRODUCTION

It is well known that a primitive substitution σ over an alphabet $\mathcal{A}$ gives rise to a non-integer numeration system, the so-called Dumont-Thomas numeration, which was presented in 1989 by Jean-Marie Dumont and Alain Thomas [16]. For each letter $x \in \mathcal{A}$ we can uniquely expand non-negative real numbers that are contained in an half-opened interval. The base is determined by the Perron-Frobenius eigenvalue induced by the substitution and the digit strings correspond to walks on a graph that is frequently called prefix-suffix graph. In the survey articles [9, 11] one finds detailed information concerning the Dumont-Thomas numeration and its position within the theory of numeration systems in English.

As main result of the present article we generalise the concept of the Dumont-Thomas numeration (Theorem 4.1). The main ingredient for achieving this are coding prescriptions. A coding prescription is, loosely spoken, a residue system with respect to a substitution (*cf.* [40]). In [41] coding prescriptions have already been used for numeration, namely, for representing integers.

At first we associate with a setting (σ, c) (that is a pair consisting of a primitive substitution σ over an alphabet $\mathcal{A}$ and a coding prescription c with respect to σ) a finite family $\mathfrak{C}_{\sigma,c} = \{I_{\sigma,c}(x) : x \in \mathcal{A} \cup \overline{A}\}$ of compact real sets, where $\overline{A}$ is the set of inverse letters. The use of inverse letters is an important tool in the entire article. This family $\mathfrak{C}_{\sigma,c}$ is given by a graph directed construction in the sense of [28]. The structure of the sets $I_{\sigma,c}(x)$ is vital for the further proceeding since these sets will serve as domain for our expansions. Not all settings have what it takes to induce a numeration system. In fact, for two concrete classes of settings, the *Continuous setting* and the *Even setting*, the set $I_{\sigma,c}(x)$ is an interval for all $x \in \mathcal{A} \cup \overline{A}$ and we can uniquely represent each element $\gamma \in \tilde{I}_{\sigma,c}(x)$ (that is $I_{\sigma,c}(x)$ with the right end point removed) as

$$\gamma = d_1\theta^{-1} + d_2\theta^{-2} + d_3\theta^{-3} + \cdots$$

where the base θ is the Perron-Frobenius eigenvalue of the substitution σ. The digit string $d_1, d_2, d_3 \dots$ is given by an infinite walk on a finite graph and consists of real numbers that also can be negative (contrary to the classical Dumont-Thomas numeration). The exact shape of the (finite) set of digits is determined by the coding prescription c.

For a special type of coding prescription that we will denote by c_+ the numeration system induced by (σ, c_+) corresponds exactly to the Dumont-Thomas numeration induced by σ. This makes the relation with the classical beta-expansion introduced by Rényi in [32] quite obvious. Indeed, it is well known that for an algebraic integer β with sofic beta-shift there exists a special *beta-substitution* σ_β such that the associated Dumont-Thomas numeration corresponds to the

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beta-expansion induced by β (see [11, 17]). Therefore, we recover the beta-expansion with respect to β by the setting (σ_β, c_+).

But our general approach also covers generalised notions of beta-expansions with respect to a domain different from the unit interval $[0,1)$. In particular, let $\delta \in [0,1)$ and $\beta > 1$. Then beta-expansions can be defined with respect to the base β and the domain $[-\delta, 1-\delta)$ in a straightforward way (see [39]). We refer to these expansions as (β,δ)-expansions. The case $\delta = 0$ yields the classical beta-expansions while for $\delta = 1/2$ we obtain the symmetric beta-expansions that have been introduced in [6]. Now, for pairs (β,δ) that satisfy special conditions there exist a substitution $\sigma_{\beta,\delta}$ and a coding prescription c_δ such that we can retrieve the respective (β,δ)-expansions by the numeration system induced by $(\sigma_{\beta,\delta}, c_\delta)$.

The above mentioned relations are the most obvious ones and we do absolutely not claim completeness. Au contraire, we strongly suggest that there are further relations with known notions of numeration. An exact characterisation of suitable settings involves several combinatorial considerations that would go beyond the constraints of the present article. We rather formulate this as open questions that hopefully will be treated in subsequent researches.

An interesting aspect concerning expansions of real numbers (with respect to algebraic bases) are finiteness and periodicity properties. The research started with the beta-expansion. In [12, 33] it was asked for which bases β the elements of $\mathbb{Q}(\beta) \cap [0,1)$ have an eventually periodic beta-expansion. For Salem numbers this question is up to now unsolved. In [19] the finiteness property was introduced for bases β that allow a finite expansion for the elements of $\mathbb{Z}[\beta^{-1}] \cap [0,1)$. In [6] and [39] the definition of finiteness has been adapted to generalised (β,δ)-expansions. An appropriate finiteness property for the Dumont-Thomas numeration was stated in [11]. Usually the verification of the finiteness property is a problem that only can be solved algorithmically (see, for example, [6, 22, 24, 35, 39]). Therefore, it is a challenge to obtain more general characterisation results.

We define appropriate finiteness and periodicity properties for our numeration systems that are compatible with the existing ones. As main result in this context we show that finiteness and periodicity do not depend on the choice of the coding prescription but only on the substitution (Theorem 4.7). As a consequence it turns out that several known notions of finiteness are equivalent. This may have an impact on respective researches.

The study of generalised notions of beta-expansions became increasingly popular in the last years. The relation with substitutions is completely new and provides synergies that may push the research forward. Apart from the finiteness property we want to explicitly mention the associated (fractal) tiles and tilings. For the classical beta-expansion these objects are quite well studied (see, *e.g.*, [2, 31]), not least because they are, in fact, Rauzy fractals induced by substitutions (see, for example, [11]). Tiles associated with variants of beta-expansions have been studied in [20, 23] and show quite unexpected properties. The current results indicate that these tiles are also related with Rauzy fractals.

If we compare substitutions induced (β,δ)-expansions with respect to the same base β but for different choices of δ then we note that they coincide up to the order of the letters, so-called flips. Especially, the substitutions have the same incidence matrix. The common dynamics of such substitutions has been studied, for instance, in [34, 36, 37]. Our results suggest that this topic can be studied from the point of view of (generalised) beta-expansions.

Observe that the Dumont-Thomas numeration has applications in the study of the topology of fractals tiles (see [3, 4, 5, 26, 27, 29]). It is straightforward to ask whether the present research can be applied here.

The article is organised in the following way. In Section 2 we introduce our formalism concerning the set of inverse letters $\overline{\mathcal{A}}$ and the words over $\mathcal{A} \cup \overline{\mathcal{A}}$. In fact, we come in contact with the free group generated by the alphabet $\mathcal{A}$. We specify the notion of a primitive substitution σ over $\mathcal{A}$ and associate to it a graph G_σ. The vertex set of this graph is $\mathcal{A} \cup \overline{\mathcal{A}}$, hence it differs from the well-known prefix-suffix graph (which we will denote by F_σ). Afterwards we define what we mean by a coding prescription c with respect to σ and relate the setting (σ, c) with a subgraph $G_{\sigma,c}$ of G_σ. In Section 3 we introduce the collection $\mathfrak{C}_{\sigma,c}$ as realisation of the subgraph $G_{\sigma,c}$. We show several general properties of the sets $I_{\sigma,c}(x)$. For two types of settings (the Continuous setting and the Even setting) $\mathfrak{C}_{\sigma,c}$ will turn out to be a family of intervals. In the further process we will concentrate on these settings. In Section 4 we state our main result, numeration systems induced by a setting (σ, c), and define the periodicity and finiteness property. We will see that finiteness and periodicity properties actually do not depend on the coding prescription. In Section 5 we will discuss the exact relation with different notions of (generalised) beta-expansions. The article is accompanied by an example that helps us to better understand the notations and visualises several results. Further illustrative examples can be found in Section 6.

2. Definitions, preliminary notations and preparation

2.1. **Free monoids and free groups.** Throughout the article we let $\mathbb{N}$ denote the set of positive integers while $\mathbb{N}^0$ is the set of non-negative integers. For an $m \in \mathbb{N}$ let $\mathcal{A} = \{1, \dots, m\}$ be an alphabet and $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$. We define the set of inverse letters $\overline{\mathcal{A}} := \{\overline{x} : x \in \mathcal{A}\}$. Analogously, $\overline{\mathcal{A}}^*$ and $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ denote the sets of finite words over $\overline{\mathcal{A}}$ and $\mathcal{A} \cup \overline{\mathcal{A}}$, respectively. We write ε for the empty word and $\mathcal{A}^+ := \mathcal{A}^* \setminus \{\varepsilon\}$ ($\overline{\mathcal{A}}^+ := \overline{\mathcal{A}}^* \setminus \{\varepsilon\}$, respectively) for the set of non-empty words over $\mathcal{A}$ ($\overline{\mathcal{A}}$, respectively).

Let $\sim$ be the equivalence relation on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the cancellation law, *i.e.* $\sim$ is the transitive hull of the relation $x\bar{x} \sim \varepsilon \sim \bar{x}x$ for all $x \in \mathcal{A}$. Then $(\mathcal{A} \cup \overline{\mathcal{A}})^*/\sim$ is the free group generated by $\mathcal{A}$. For two words $X, X' \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $X \sim X'$ we write $X = X'$ (modulo $\sim$). Observe that for convenience we will skip the term "(modulo $\sim$)" in large parts of the article since there is no danger of confusion.

An inverse modulo $\sim$ of a word $X = x_1 \cdots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ is given by

$$\overline{X} := \bar{x}_n \, \bar{x}_{n-1} \cdots \bar{x}_1,$$

where $\overline{\overline{x}} = x$ for all $x \in \mathcal{A}$. For $y \in \mathcal{A}$ and $X = x_1, \dots, x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ we define

$$|X|_y := \#\{j \in \{1, \dots, n\} : x_j = y\} - \#\{j \in \{1, \dots, n\} : x_j = \overline{y}\},$$

that is the difference of the number of occurrences of y and the number of occurrences of $\bar{y}$ in X. Let

$$|X| := \sum_{y \in \mathcal{A}} |X|_y,$$
$$\mathbf{l}(X) := (|X|_1, |X|_2, \dots, |X|_m)^T \in \mathbb{Z}^m.$$

Observe that these definitions are compatible with $\sim$ and behave additively with respect to the concatenation of words, *i.e.* for all $X, X', Y \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $X = X'$ (modulo $\sim$), and $y \in \mathcal{A}$ we have $|XY|_y = |X|_y + |Y|_y$ and $|X|_y = |X'|_y$. This immediately implies that $|XY| = |X| + |Y|$, $|X| = |X'|$ as well as $\mathbf{l}(XY) = \mathbf{l}(X) + \mathbf{l}(Y)$, $\mathbf{l}(X) = \mathbf{l}(X')$ also hold.

We define the partial ordering $\le$ on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ by

$$X \le Y \iff \overline{X}Y \in \mathcal{A}^* \quad \text{(modulo } \sim).$$

The corresponding strict partial ordering $<$ is given by

$$X < Y \iff \overline{X}Y \in \mathcal{A}^+ \quad \text{(modulo } \sim).$$

Note that $\le$ is a generalisation of the prefix and suffix property of words. Indeed, for $X, Y \in \mathcal{A}^*$ we have $X \le Y$ if and only if X is a prefix of Y and $\overline{Y} \le \overline{X}$ if and only if X is a suffix of Y. The strict version $<$ means that the prefix and suffix, respectively, is proper. One easily verifies that for words $X', Y' \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $X = X'$ and $Y = Y'$ (modulo $\sim$) we have $X \le Y$ if and only if $X' \le Y'$, hence $\le$ can be transferred to a partial ordering on the free group $(\mathcal{A} \cup \overline{\mathcal{A}})^*/\sim$.

Observe that all the words over $\mathcal{A} \cup \overline{\mathcal{A}}$ that appear in the present paper are contained in $\mathcal{A}^* \cup \overline{\mathcal{A}}^*$ (modulo $\sim$). Therefore, in the rest of the article we can represent all words by elements of $\mathcal{A}^* \cup \overline{\mathcal{A}}^*$.

2.2. **Substitutions and related graphs.** Let σ be a substitution over $\mathcal{A}$, that is a morphism of the free monoid $\mathcal{A}^*$. Throughout the article we require σ to be primitive which means that there exists a positive integer n such that we have $|\sigma^n(y)|_x > 0$ for each two letters $x, y \in \mathcal{A}$. We let $\mathbf{M}_\sigma := (|\sigma(y)|_x)_{1 \le x, y \le m}$ denote the incidence matrix of σ. Observe that the primitivity of σ implies that $\mathbf{M}_\sigma$ is a primitive matrix and, thus, it possesses a dominant real Perron-Frobenius eigenvalue $\theta > 1$.

A substitution σ extends in a natural way to words over $\mathcal{A} \cup \overline{\mathcal{A}}$ by setting $\sigma(\bar{x}) := \overline{\sigma(x)}$ for each $x \in \mathcal{A}$. Note that for $X, Y \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ with $X = Y$ (modulo $\sim$) we have $\sigma(X) = \sigma(Y)$ (modulo $\sim$). This makes σ a morphism of the free group $(\mathcal{A} \cup \overline{\mathcal{A}})^*/\sim$. Furthermore, for each $X \in (A \cup \overline{\mathcal{A}})^*$ we have

$$\mathbf{l}(\sigma(X)) = \mathbf{M}_\sigma \mathbf{l}(X). \tag{2.1}$$

Note that we are not confronted with the problem of uncontrolled cancellation which is a great challenge in the study of the dynamics of (general) endomorphisms of the free group (*cf.* [8]).

We associate with the substitution σ the directed multigraph G_σ with set of vertices $\mathcal{A} \cup \overline{\mathcal{A}}$. Whenever $\sigma(x) = PyS$ for $x, y \in \mathcal{A}$ and $P, S \in \mathcal{A}^*$ there are four edges in G_σ

- an edge from x to y labelled by (P, y);
- an edge from x to $\bar{y}$ labelled by $(Py, \bar{y})$;
- an edge from $\bar{x}$ to y labelled by $(\overline{yS}, y)$;
- an edge from $\bar{x}$ to $\bar{y}$ labelled by $(\overline{S}, \bar{y})$.

Let $G_\sigma^1(x)$ denote the set of edges that start in x represented by their labels and observe that for each $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ the set $G_\sigma^1(x)$ contains exactly one edge (D, y) with $D = \varepsilon$ which we will refer to as zero-edge.

If $(D, x_1) \in G_\sigma^1(x)$ then, by definition, D is a prefix of $\sigma(x)$ or $\overline{D}$ is a suffix of $\sigma(\bar{x})$. Therefore, if $(D', x_1') \neq (D, x_1)$ is another edge that starts in x then D and D' are $\prec$-comparable: we have $D_1 \prec D_1'$, $D_1' \prec D_1$ or $D_1 = D_1'$. We see by definition that in the latter case $x_1 \in \mathcal{A}$ and $x_1' \in \overline{\mathcal{A}}$ (or vice versa), and the edges are different from the zero edge. Motivated by this observation we extend the ordering $\prec$ to the set $G_\sigma^1(x)$ of edges and define

$$(D, x_1) \prec (D', x_1') \iff \begin{cases} D \prec D' & \text{if } D \neq D' \\ Dx_1 \prec D'x_1' & \text{if } D = D' \end{cases}$$

Due to primitivity the graph G_σ is strongly connected. Observe that G_σ is a generalisation of the well-known prefix-suffix graph F_σ (see, e.g., [15]) defined as follows. The set of vertices of F_σ is $\mathcal{A}$ and there is an edge from x to y in F_σ labelled by (P, y, S) whenever $\sigma(x) = PyS$ (with $P, S \in \mathcal{A}^*$). We see that $(P, y, S) \in F_\sigma^1(x)$ if and only if $(P, y) \in G_\sigma^1(x)$.

Example (Accompanying example). We illustrate our formalism and our results by an example that will accompany us until Section 4. Further examples (that refer especially to the results of Section 5) can be found in Section 6.

Let $\sigma : 1 \mapsto 112, 2 \mapsto 1$ over the alphabet $\mathcal{A} = \{1, 2\}$. We find the associated graph G_σ in Figure 1 (left). For comparison the classical prefix-suffix graph F_σ is depicted on the right hand side. The set of edges that start, for instance, in $\bar{1}$ is given by

$$G_\sigma^1(\bar{1}) = \{(\overline{112}, 1), (\overline{12}, \bar{1}), (\overline{12}, 1), (\bar{2}, \bar{1}), (\bar{2}, 2), (\varepsilon, \bar{2})\}.$$

The latter one is the zero-edge. Observe that we have

$$(\overline{112}, 1) \prec (\overline{12}, \bar{1}) \prec (\overline{12}, 1) \prec (\bar{2}, \bar{1}) \prec (\bar{2}, 2) \prec (\varepsilon, \bar{2}).$$

For a positive integer n a path of length n on G_σ is a sequence of edges $(D_j, x_j)_{j=1}^n$ such that $(D_j, x_{j+1}) \in G_\sigma^1(x_j)$ holds for all $j \in \{1, \dots, n-1\}$. We say that the path starts in $x \in \mathcal{A} \cup \overline{A}$ if $(D_1, x_1) \in G_\sigma^1(x)$. We write $G_\sigma^n(x)$ for the set of paths of length n that start in x. The uniquely determined element of $G_\sigma^n(x)$ that consists of zero edges only is the zero path of length n (that

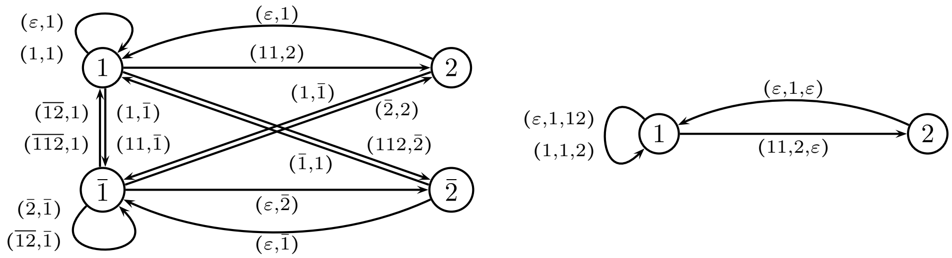

FIGURE 1. Left: The graph G_σ associated with the substitution $\sigma : 1 \mapsto 112, 2 \mapsto 1$. Right: The corresponding prefix-suffix graph F_σ. For not getting lost in a jumble of edges we joined edges with the same origin and the same destination and put multiple labels in a respective way.

starts in x). We extend the ordering on the edges in a lexicographical way to the elements of $G_\sigma^n(x)$. In particular, for $(D_j, x_j)_{j=1}^n, (D'_j, x'_j)_{j=1}^n \in G_\sigma^n(x)$ we have

$$(D_j, x_j)_{j=1}^n \prec_{\text{lex}} (D'_j, x'_j)_{j=1}^n \iff \exists n_0 \in \{1, \dots, n\} : (D_j, x_j) = (D'_j, x'_j) \text{ for all } j < n_0 \wedge (D_{n_0}, x_{n_0}) \prec (D'_{n_0}, x'_{n_0}).$$

For $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ consider a path of length two that starts in x, hence $(D_1, x_1)(D_2, x_2) \in G_\sigma^2(x)$. Then trivial calculations yield that $(\sigma(D_1)D_2, x_2) \in G_{\sigma^2}^1(x)$, that is $(\sigma(D_1)D_2, x_2)$ is an edge in the graph associated with σ^2 with initial vertex x. Especially, we see that $\sigma(D_1)D_2$ is (modulo $\sim$) contained in $\mathcal{A}^* \cup \overline{\mathcal{A}}^*$.

On the other hand, for each edge $(D, x_1) \in G_{\sigma^2}^1(x)$ there exists a path $(D_1, x'_1)(D_2, x_1) \in G_\sigma^2(x)$ such that $D = \sigma(D_1)D_2$ (in fact, there are exactly two paths with this property). Observe that this composition of edges is in general not order preserving, that is if $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $(D_j, x_j)_{j=1}^2, (D'_j, x'_j)_{j=1}^2 \in G_\sigma^2(x)$ with $(D_j, x_j)_{j=1}^2 \prec_{\text{lex}} (D'_j, x'_j)_{j=1}^2$ then we cannot conclude that $(\sigma(D_1)D_2, x_2) \prec (\sigma(D'_1)D'_2, x'_2)$.

The composition of edges extends to paths of arbitrary length n: if $(D_j, x_j)_{j=1}^n \in G_\sigma^n(x)$ then $(D, x_n) \in G_{\sigma^n}^1(x)$ where

$$D := \sigma^{n-1}(D_1)\sigma^{n-2}(D_2)\cdots\sigma(D_{n-1})D_n \in \mathcal{A}^* \cup \overline{\mathcal{A}}^*$$

and for each edge in $G_{\sigma^n}^1(x)$ we always can find a corresponding path of length n in $G_\sigma^n(x)$.

An (infinite) walk on G_σ is an infinite sequence of edges $(D_j, x_j)_{j\geq 1}$ such that $(D_j, x_j)_{j\geq 1}^n$ is a path on G_σ for each $n \geq 1$. The walk starts in $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ if $(D_1, x_1) \in G_\sigma^1(x)$. The expression $G_\sigma^\infty(x)$ represents the set of walks that start in x. In this article infinite walks are denoted by Gothic letters. We extend the ordering $\preceq_{\text{lex}}$ in a straightforward way to the elements of $G_\sigma^\infty(x)$. In particular for $\mathfrak{x}, \mathfrak{x}' \in G_\sigma^\infty(x)$ with $\mathfrak{x} = (D_j, x_j)_{j\geq 1}$ and $\mathfrak{x}' = (D'_j, x'_j)_{j\geq 1}$ we have

$$\mathfrak{x} \preceq_{\text{lex}} \mathfrak{x}' \iff \forall n \geq 1 : (D_j, x_j)_{j=1}^n \preceq_{\text{lex}} (D'_j, x'_j)_{j=1}^n,$$
$$\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}' \iff \mathfrak{x} \preceq_{\text{lex}} \mathfrak{x}' \wedge \mathfrak{x} \neq \mathfrak{x}'.$$

In accordance with our notations we call the uniquely determined element of $G_\sigma^\infty(x)$ that consists of zero edges only the *zero walk that starts in x*.

Recall that we consider primitive substitutions and, thus, the incidence matrix $\mathbf{M}_\sigma$ has a dominant real eigenvalue that we denoted by θ. Fix a left eigenvector $\mathbf{v}$ of $\mathbf{M}_\sigma$ with respect to θ which we may assume to be strictly positive. Define

$$\lambda : (\mathcal{A} \cup \overline{A})^* \longrightarrow \mathbb{R}, X \longmapsto \langle \mathbf{l}(X), \mathbf{v} \rangle .$$

Note that λ is compatible with $\sim$ and the positivity of $\mathbf{v}$ implies λ to be order preserving, that is for $X, X' \in (\mathcal{A} \cup \overline{A})^*$ we have $X = X'(\text{modulo} \sim) \Rightarrow \lambda(X) = \lambda(X')$ and $X \prec X' \Rightarrow \lambda(X) < \lambda(X')$. Observe that $\mathbf{v} = (\lambda(1), \dots, \lambda(m))$.

Now let $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $\mathfrak{x} := (D_j, x_j)_{j\geq 1} \in G_\sigma^\infty(x)$. Consider the series

$$\Lambda(\mathfrak{x}) := \sum_{j\geq 1} \theta^{-j}\lambda(D_j) \tag{2.2}$$

which can be immediately seen to converge since G_σ is a finite graph and $\theta > 1$. Thus, Λ is a well-defined function that assigns to each infinite walk on G_σ a real number. Observe that Λ depends on the choice of $\mathbf{v}$ (as λ does) but usually $\mathbf{v}$ is fixed in the respective context and does not vary.

Example (Accompanying example). We want to illustrate the latter notations by our example. We have

$$\mathbf{M}_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, the dominant root of $\mathbf{M}_\sigma$ is $\theta = 1+\sqrt{2}$. As a left eigenvector with respect to θ we fix $\mathbf{u} := (\theta, 1)$. Hence, $\lambda(1) = \theta$ and $\lambda(2) = 1$. We are interested in elements of the set $G_\sigma^\infty(\bar{1})$, that is the set of walks that start in the vertex $\bar{1}$. The zero walk is given by $\mathfrak{x}_0 = \big((\varepsilon, \bar{2}), (\varepsilon, \bar{1})\big)^\omega$. Further examples of walks that start in $\bar{1}$ are $\mathfrak{x}_1 = (\bar{2}, \bar{1})^\omega$, $\mathfrak{x}_2 = \big((\bar{2}\bar{1}, 1)(11, \bar{1})\big)^\omega$ and $\mathfrak{x}_3 = (\bar{2}\bar{1}, 1)\left((11, 2)(\varepsilon, 1)\right)^\omega$. We clearly have $\Lambda(\mathfrak{x}_0) = 0$. Furthermore, we obtain

$$\begin{aligned} \Lambda(\mathfrak{x}_1) &= \sum_{j\geq 1} \lambda(\bar{2})\theta^{-j} = \sum_{j\geq 1} -1\cdot\theta^{-j} = -(\theta-1)^{-1} = -\sqrt{2}/2, \\ \Lambda(\mathfrak{x}_2) &= \sum_{j\geq 1} \lambda(\bar{2}\bar{1})\theta^{-2j+1} + \sum_{j\geq 1} \lambda(11)\theta^{-2j} = -\sqrt{2}/2, \\ \Lambda(\mathfrak{x}_3) &= \lambda(\bar{2}\bar{1})\theta^{-1} + \sum_{j\geq 1} \lambda(11)\theta^{-2j} = 1-\sqrt{2}. \end{aligned}$$

2.3. **Coding prescriptions and subgraphs.** In the present article we are mainly interested in subgraphs of G_σ induced by so-called coding prescriptions.

Definition 2.1 (Coding prescription, *cf.* [40])**.** Let σ be a primitive substitution over the alphabet $\mathcal{A}$. A *coding prescription (with respect to σ)* is a function c that assigns to each element of $\mathcal{A} \cup \overline{\mathcal{A}}$ a finite set of integers such that for each $x \in \mathcal{A}$ we have

(1) $c(x) \subset \{0, \dots, |\sigma(x)| - 1\}$ and $c(\bar{x}) \subset \{1 - |\sigma(x)|, \dots, 0\}$;
(2) for each $k \in \{0, \dots, |\sigma(x)|\}$ we have $k \in c(x)$ if and only if $k - |\sigma(x)| \notin c(\bar{x})$.

If c is a coding prescription with respect to σ then we call the pair (σ, c) a setting.

From the definition we can deduce several basic but useful facts that we keep in mind.

- 0 is contained in both $c(x)$ and $c(\bar{x})$;
- $\#c(x) + \#c(\bar{x}) = |\sigma(x)| + 1$;
- $c(x) \cup c(\bar{x})$ is a complete residue system modulo $|\sigma(x)|$;
- the sets $c(x)$ and $|\sigma(x)| + c(\bar{x})$ form a partition of the set $\{0, \dots, |\sigma(x)|\}$;
- c is completely determined by fixing $c(x)$ for all $x \in \mathcal{A}$.

For a setting (σ, c) we define by $G_{\sigma,c}$ the spanning subgraph of G_σ such that for each $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ we have

$$G_{\sigma,c}^1(x) = \left\{(D, x_1) \in G_\sigma^1(x) : |D| \in c(x)\right\}.$$

Observe that the set $G_{\sigma,c}^1(x)$ of edges that start in x is a $\prec$-chain, hence there exists a minimal element (minimal edge) (D^-, x_1^-), and a maximal element (maximal edge) (D^+, x_1^+). If $x \in \mathcal{A}$ then the minimal element is the zero edge and $x_1^+ \in \mathcal{A}$ while for $x \in \overline{\mathcal{A}}$ the maximal element is the zero edge and $x_1^- \in \overline{\mathcal{A}}$. Depending on the coding prescription c, the minimal edge and the maximal edge may coincide.

Example (Accompanying example). We continue with our example $\sigma : 1 \mapsto 112, 2 \mapsto 1$. There are 4 coding prescriptions with respect to σ.

$$
\begin{array}{llll}
c_1: & 1 \mapsto \{0\} & 2 \mapsto \{0\} & (\Longrightarrow \bar{1} \mapsto \{-2,-1,0\}, \bar{2} \mapsto \{0\}) \\
c_2: & 1 \mapsto \{0,1\} & 2 \mapsto \{0\} & (\Longrightarrow \bar{1} \mapsto \{-1,0\}, \bar{2} \mapsto \{0\}) \\
c_3: & 1 \mapsto \{0,1,2\} & 2 \mapsto \{0\} & (\Longrightarrow \bar{1} \mapsto \{0\}, \bar{2} \mapsto \{0\}) \\
c_4: & 1 \mapsto \{0,2\} & 2 \mapsto \{0\} & (\Longrightarrow \bar{1} \mapsto \{-2,0\}, \bar{2} \mapsto \{0\})
\end{array}
$$

Each of them determines a setting and a certain subgraph of G_σ (see Figure 2). Consider again the

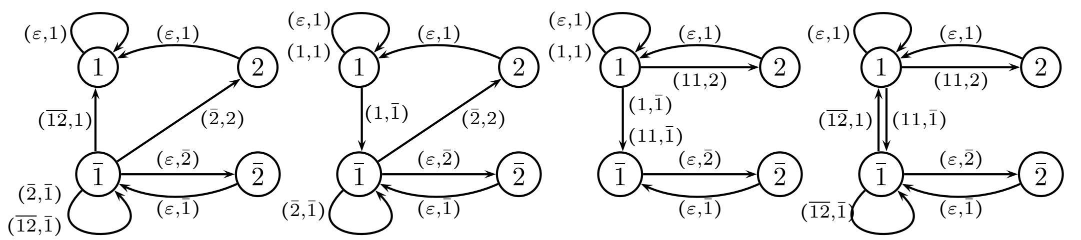

FIGURE 2. The 4 subgraphs of G_σ for our accompanying example. From left to right: G_{σ,c_1}, G_{σ,c_2}, G_{σ,c_3}, G_{σ,c_4}.

edges that start in the vertex $\bar{1}$. We have $G^1_{\sigma,c_1}(\bar{1}) = \{(\overline{12},\bar{1}),(\overline{12},1),(\bar{2},\bar{1}),(\bar{2},2),(\varepsilon,\bar{2})\}$, $G^1_{\sigma,c_2}(\bar{1}) = \{(\bar{2},\bar{1}),(\bar{2},2),(\varepsilon,\bar{2})\}$, $G^1_{\sigma,c_3}(\bar{1}) = \{(\varepsilon,\bar{2})\}$, $G^1_{\sigma,c_4}(\bar{1}) = \{(\overline{12},\bar{1}),(\overline{12},1),(\varepsilon,\bar{2})\}$. The maximal edge is the zero-edge $(\varepsilon,\bar{2})$ for all four setting. The minimal edge is given by $(\overline{12},\bar{1})$ for the settings (σ,c_1) and (σ,c_4), respectively, and $(\bar{2},\bar{1})$ for the settings (σ,c_2). For the setting (σ,c_3) the minimal edge and the maximal edge coincide.

Based on our previous notations we let $G^n_{\sigma,c}(x)$ denote the set of paths of length n on $G_{\sigma,c}$ that start in x while $G^\infty_{\sigma,c}(x)$ is the set of infinite walks on $G_{\sigma,c}$ that start in x. Observe that the zero path of length n (zero walk, respectively) is always an element of $G^n_{\sigma,c}(x)$ ($G^\infty_{\sigma,c}(x)$, respectively). Furthermore, $G^n_{\sigma,c}(x)$ contains a maximal path (minimal path, respectively) of length n, and $G^\infty_{\sigma,c}(x)$ contains the maximal walk (minimal walk, respectively), which are the maximal (minimal, respectively) elements with respect to $\preceq_{\text{lex}}$ and consist of maximal (minimal, respectively) edges only. Depending on whether $x \in \mathcal{A}$ or $x \in \overline{\mathcal{A}}$, either the minimal or the maximal path (walk, respectively) coincides with the zero path (walk, respectively).

The finite paths on the subgraphs $G_{\sigma,c}$ have already been studied in [41]. It turned out that here the composition of edges is an injective operation that yields a coding prescription with respect to a higher power of σ. The most important facts are collected in the next proposition.

Proposition 2.2. *Let σ be a primitive substitution over the alphabet $\mathcal{A}$, c a coding prescription with respect to σ and n a positive integer. Then the following items hold.*

(1) The function $c^{(n)}$ with domain $\mathcal{A} \cup \overline{\mathcal{A}}$ defined by

$$c^{(n)} : x \longmapsto \left\{ \sum_{j=1}^{n} \left|\sigma^{n-j}(D_j)\right| : (D_j, x_j)_{j=1}^n \in G^n_{\sigma,c}(x) \right\}$$

is a coding prescription with respect to σ^n.

(2) The set of edges in the graph $G_{\sigma^n,c^{(n)}}$ that start in x is given by

$$G^1_{\sigma^n,c^{(n)}}(x) = \left\{ \left(\sigma^{n-1}(D_1)\cdots\sigma(D_{n-1})D_n, x_n\right) : (D_j, x_j)_{j=1}^n \in G^n_{\sigma,c}(x) \right\}.$$

(3) For each $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and distinct paths $(D_j,x_j)_{j=1}^n, (D'_j,x'_j)_{j=1}^n \in G^n_{\sigma,c}(x)$ we have $(D,x_n) \neq (D',x'_n)$ where $D = \sigma^{n-1}(D_1)\cdots\sigma(D_{n-1})D_n$ and $D' = \sigma^{n-1}(D'_1)\cdots\sigma(D'_{n-1})D'_n$.

Proof. The notations from [41] differ a little from the actual ones. Coding prescriptions are defined as the function that assigns to a pair of letters $ab \in \mathcal{A}^2$ the set $c(\bar{a}) \cup c(b)$. The associated graph $H_{\sigma,c}$ has vertex set $\mathcal{A}^2$ and each edge in $H_{\sigma,c}$ corresponds to two edges in $G_{\sigma,c}$. In particular, let $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $(D,x_1) \in G^1_{\sigma,c}(x)$. If (D,x_1) is not the zero edge then there exists exactly one further edge $(D, x_1') \in G^1_{\sigma,c}(x)$ and we either have $x_1 \in \mathcal{A}$, $x_1' \in \overline{\mathcal{A}}$ or vice versa. If (D, x_1) is the zero edge then there is no other similar edge. This shows that if $ab \in \mathcal{A}^2$ and $(D, x_1) \in G^1_{\sigma,c}(\bar{a}) \cup G^1_{\sigma,c}(b)$ then there always exists exactly one further edge (D, x_1') (with either $x_1 \in \mathcal{A}$, $x_1' \in \overline{\mathcal{A}}$ or vice versa). Now, there is an edge from $ab \in \mathcal{A}^2$ to $a_1 b_1 \in \mathcal{A}^2$ labelled by $(D, a_1 b_1)$ if $(D, \bar{a}_1)$ and (D, b_1) are contained in $G^1_{\sigma,c}(\bar{a}) \cup G^1_{\sigma,c}(b)$. This implies that the set of edges that start in the vertex ab is given by

$$H^1_{\sigma,c}(ab) = \left\{(D, a_1 b_1) : (D, \bar{a}_1), (D, b_1) \in G^1_{\sigma,c}(\bar{a}) \cup G^1_{\sigma,c}(b)\right\}.$$

With these preparations one easily obtains Item (1) from [41, Theorem 2.4], Item (2) from [41, Corollary 3.4], and Item (3) by observing [41, Remark 3.3]. □

Remark 2.3. The graph $H_{\sigma,c}$, as defined in the proof of Proposition 2.2, has m^2 vertices. For showing the results in this article the smaller graph $G_{\sigma,c}$ ($2|\mathcal{A}| = 2m$ vertices) is much more convenient. In Corollary 4.2 we formulate our main Theorem in terms of the graph $H_{\sigma,c}$ which yields some interesting aspects.

We introduce two special coding prescriptions c_- and c_+ with respect to a primitive substitution σ that assign either to all elements of $\mathcal{A}$ or to all elements of $\overline{\mathcal{A}}$ the set $\{0\}$:

$$\begin{aligned} c_- :\quad & x \in \mathcal{A} \longmapsto \{0\}, & x \in \overline{\mathcal{A}} \longmapsto \{-|\sigma(x)| + 1, \dots, 0\}, \\ c_+ :\quad & x \in \mathcal{A} \longmapsto \{0, \dots, |\sigma(x)| - 1\}, & x \in \overline{\mathcal{A}} \longmapsto \{0\}. \end{aligned}$$

Observe that G_{σ,c_+} is closely related with the prefix-suffix graph since for each $x \in \mathcal{A}$ we have

$$F^1_\sigma(x) = \left\{(P, x_1, S) : (P, x_1) \in G^1_{\sigma,c_+}(x), S = \overline{Px_1}\sigma(x)\right\}.$$

In Theorem 4.1 we will see that a setting (σ, c_+) corresponds to the Dumont-Thomas numeration.

We already observed that the composition of edges is in general not order preserving. We introduce the ordering condition (O) for a setting (σ, c) by

$$\text{(O)} \quad \forall n \geq 1, x \in \mathcal{A}, (D_j, x_j)_{j=1}^n, (D_j', x_j')_{j=1}^n \in G^n_{\sigma,c}(x) :$$
$$(D_j, x_j)_{j=1}^n \prec_{\text{lex}} (D_j', x_j')_{j=1}^n \Longrightarrow \left(\sigma^{n-1}(D_1)\cdots\sigma(D_{n-1})D_n, x_n\right) \prec \left(\sigma^{n-1}(D_1')\cdots\sigma(D_{n-1}')D_n', x_n'\right).$$

The condition will turn out to be very important for our research. The following types of settings are of special interest since they satisfy (O).

Continuous setting: Here $c(x)$ is a set of consecutive integers for all $x \in \mathcal{A} \cup \overline{\mathcal{A}}$. By the definition of coding prescriptions this is equivalent to

$$\text{(CS)} \qquad \forall x \in \mathcal{A} : \max c(x) - \min c(\bar{x}) = |\sigma(x)| - 1.$$

Even setting: In this setting σ assigns to each letter a word of odd length and c assigns to each $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ a subset of $2\mathbb{Z}$. Formally we can specify this by the following condition:

$$\text{(ES)} \qquad \begin{aligned} &\forall x \in \mathcal{A} : |\sigma(x)| \equiv 1 \pmod 2 \wedge \\ &\qquad c(x) = \{0, 2, \dots, |\sigma(x)| - 1\} \quad (\Longrightarrow c(\bar{x}) = \{-|\sigma(x)| + 1, -|\sigma(x)| + 3, \dots, 0\}). \end{aligned}$$

Clearly, settings that involve c_+ or c_- satisfy (CS). Observe that if $c(x)$ is a set of consecutive integers then the definition of coding prescriptions immediately implies that the same holds for $c(\bar{x})$.

Proposition 2.4. *Let σ be a substitution over the alphabet $\mathcal{A}$ and c a coding prescription with respect to σ such that the pair (σ, c) fulfils* (CS) *or* (ES)*. Then the ordering condition* (O) *is satisfied. Furthermore, for each positive integer n the pair $(\sigma^n, c^{(n)})$ also fulfils* (CS) *or* (ES)*, respectively.*

Proof. The first part is shown in [41, Proposition 2.7]. For the second part see [41, Theorem 2.4] and [41, Lemma 3.10], respectively. □

We also have a converse statement, namely that the Continuous setting and the Even setting are the only ones that satisfy (O).

Proposition 2.5. *If a setting* (σ, c) *has the ordering property* (O) *then it either satisfies* (CS) *or* (ES).

Proof. We show that if (σ, c) has the ordering property (O) and does not satisfy (ES) then it satisfies (CS).

If (ES) is not satisfied then there exists a letter $x \in \mathcal{A}$ such that $c(x)$ or $c(\bar{x})$ contain two consecutive integers $k, k+1$. We suppose that $k \geq 0$, that is $k, k+1 \in c(x)$ (the case $k+1 \leq 0$ runs analogously). Let $P, S \in \mathcal{A}^*$, $y \in \mathcal{A}$ such that $\sigma(x) = PyS$ and $|P| = k$. By definition $G^1_{\sigma,c}(x)$ contains the edges (P, y) and $(Py, \bar{y})$.

We show indirectly that $c(y)$ (and hence $c(\bar{y})$) is a set of consecutive integers. Indeed, suppose that this claim did not hold. Then there exists a positive integer $k_1 \in c(y)$ such that $k_1 - 1 \notin c(y)$ (in fact, $k_1 \geq 2$). By definition $k_1 - 1 - |\sigma(y)| \in c(\bar{y})$. Let $(D_1, x_1) \in G^1_{\sigma,c}(y)$ and $(D_1', x_1') \in G^1_{\sigma,c}(\bar{y})$ be respective edges, i.e. $|D_1| = k_1$ and $|D_1'| = k_1 - 1 - |\sigma(y)|$. In this way we obtain two paths of length 2 that start in x: $(P, y)(D_1, x_1)$ and $(Py, \bar{y})(D_1', x_1')$ where the first one is clearly smaller then the second one with respect to $\prec$. However, we have

$$|\sigma(Py)D_1'| = |\sigma(P)| + |\sigma(y)| + k_1 - 1 - |\sigma(y)| < |\sigma(P)| + k_1 = |\sigma(P)D_1|$$

which implies that $(\sigma(Py)D_1', x_1) \prec (\sigma(P)D_1, x_1)$. This contradicts (O).

Now observe that by definition $c^{(2)}(x)$ contains the elements of $|\sigma(P)| + c(y)$ and $|\sigma(Py)| + c(\bar{y})$ which implies that $\{|\sigma(P)|, |\sigma(P)| + 1, \ldots, |\sigma(P)| + |\sigma(y)|\} \subset c^{(2)}(x)$. We therefore can repeat the argumentation from above and show that for each letter y' that occurs at position $|\sigma(P)| + 1$ to $|\sigma(Py)|$ in $\sigma^2(x)$ we also have that $c(y')$ is a set of consecutive integers. By observing that these letters are exactly the letters that occur in $\sigma(y)$ and that σ is primitive we conclude that (CS) must hold. □

Example (Accompanying example). The coding prescriptions c_1 and c_3 correspond to the special coding prescriptions c_- and c_+, respectively. The settings (σ, c_1), (σ, c_2) and (σ, c_3) satisfy (CS) while (σ, c_4) satisfies (ES). Hence, all four settings have the ordering condition (O). Observe that the graph H_{σ,c_2} is depicted in the centre of Figure 5.

Let us consider $\sigma^2 : 1 \mapsto 1121121, 2 \mapsto 112$. There are $2^6 \cdot 2^2 = 256$ different coding prescriptions with respect to σ^2. Four of them can be obtained as powers of the coding prescriptions c_1, c_2, c_3 and c_4 (due to Theorem 2.2). For instance, by considering the paths of length 2 of G_{σ,c_2} we easily calculate that $c_2^{(2)} : 1 \mapsto \{0,1,2,3\}, 2 \mapsto \{0,1\}$. The setting $\left(\sigma^2, c_2^{(2)}\right)$ also satisfies (CS) (*cf.* Proposition 2.4).

Coding prescriptions with respect to σ^2 that are not powers of coding prescriptions with respect to σ are, for example,

$$c' : 1 \mapsto \{0,1\}, 2 \mapsto \{0,1,2\},$$
$$c'' : 1 \mapsto \{0,3,4\}, 2 \mapsto \{0,1\}.$$

The setting (σ^2, c') clearly satisfies (CS) while (σ^2, c'') does neither satisfy (CS) nor (ES). In fact, only $7 \cdot 3 = 21$ of the 256 coding prescriptions with respect to σ^2 induce a Continuous setting and only one, namely $c_4^{(2)}$, induces an Even setting. The 234 remaining settings violate the ordering condition (O).

2.4. **The Dumont-Thomas numeration.**

Let σ be a primitive substitution and consider the prefix-suffix graph F_σ.

Theorem 2.6 (Dumont-Thomas numeration, *cf* [9, 11, 16])**.** *Let* $x \in \mathcal{A}$ *be an arbitrary vertex of* F_σ. *Then for each* $\gamma \in [0, \lambda(x))$ *there exists a unique walk* $(P_j, x_j, S_j)_{j\geq 1} \in F_\sigma^\infty(x)$, *such that* $S_j \neq \varepsilon$ *for infinitely many indices* $j \in \mathbb{N}$, *that satisfies*

$$\gamma = \sum_{j \geq 1} \theta^{-j} \lambda(P_j). \tag{2.3}$$

Following [11] we call the representation (2.3) the (σ, x)-*expansion* of γ. The sequence $\mathfrak{d}_{\sigma,x}(\gamma) := (\lambda(P_j))_{j\geq 1}$ is the respective digit string. We say that the Dumont-Thomas numeration induced by

σ fulfils the finiteness condition if for all $x \in \mathcal{A}$ the (σ, x)-expansion of each element of $[0, \lambda(x)) \cap \mathbb{Z}[\mathbf{v}]$ is a finite sum, where $\mathbb{Z}[\mathbf{v}] = \mathbb{Z}[\lambda(1), \ldots, \lambda(m)]$.

The main purpose of this article is to generalise the Dumont-Thomas numeration. Before we are able to properly state this in Theorem 4.1 we need the results of the next section. It will turn out that only settings that have the ordering property (O) come into question.

3. Infinite walks

Throughout the section let σ denote a primitive substitution and c a coding prescription with respect to σ. The main object that we deal with are infinite walks on G_σ and the subgraph $G_{\sigma,c}$.

For $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ let

$$I_{\sigma,c}(x) := \left\{\Lambda(\mathfrak{r}) : \mathfrak{r} \in G^\infty_{\sigma,c}(x)\right\} \subset \mathbb{R}.$$

We associate with the setting (σ, c) the finite collection

$$\mathfrak{C}_{\sigma,c} := \{I_{\sigma,c}(x) : x \in \mathcal{A} \cup \overline{\mathcal{A}}\}.$$

If $\mathfrak{r} = (D_j, x_j)_{j\ge1} \in G^\infty_\sigma(x)$ and $(D, x) \in G^1_\sigma(x')$ then we obviously have that

$$\mathfrak{r}' = (D, x)(D_1, x_1)(D_2, x_2)(D_3, x_3)\ldots \in G^\infty_\sigma(x')$$

and $\Lambda(\mathfrak{r}') = \theta^{-1}(\lambda(D) + \Lambda(\mathfrak{r}))$. Therefore,

$$G^\infty_\sigma(x) = \bigcup_{(D_1,x_1)\in G^1_\sigma(x)} \{(D_j, x_j)_{j\ge1} : (D_j, x_j)_{j\ge2} \in G^\infty_\sigma(x_1)\},$$

$$G^\infty_{\sigma,c}(x) = \bigcup_{(D_1,x_1)\in G^1_{\sigma,c}(x)} \{(D_j, x_j)_{j\ge1} : (D_j, x_j)_{j\ge2} \in G^\infty_{\sigma,c}(x_1)\},$$

and especially

$$I_{\sigma,c}(x) = \bigcup_{(D_1,x_1)\in G^1_{\sigma,c}(x)} \theta^{-1}\left(\lambda(D_1) + I_{\sigma,c}(x_1)\right). \tag{3.1}$$

From this we see that our construction fits into the framework of graph directed iterated function systems (GIFS) presented in [28]. In particular, when we associate with each edge (D, x_1) that appears in $G_{\sigma,c}$ the similarity

$$f_D : \mathbb{R} \longrightarrow \mathbb{R}, \xi \longmapsto \theta^{-1}(\xi + \lambda(D))$$

(with ratio $\theta^{-1} < 1$) then we obtain a realisation of $G_{\sigma,c}$ and $\mathfrak{C}_{\sigma,c}$ is the uniquely determined invariant set list. We immediately deduce that $\mathfrak{C}_{\sigma,c}$ is a collection of compact sets. The main result of the present section is a characterisation of $\mathfrak{C}_{\sigma,c}$. We denote by μ the one-dimensional Lebesgue measure.

Theorem 3.1. *Let σ be a primitive substitution over $\mathcal{A}$ and c a coding prescription with respect to σ. Then the following items hold.*

(i) For each $x \in \mathcal{A}$ we have

$$I_{\sigma,c}(x) \subseteq [0, \Lambda(\mathfrak{r}^+)] \subseteq [0, \lambda(x)],$$
$$I_{\sigma,c}(\bar{x}) \subseteq [\Lambda(\mathfrak{r}^-), 0] \subseteq [-\lambda(x), 0],$$

where $\mathfrak{r}^+ \in G^\infty_{\sigma,c}(x)$ and $\mathfrak{r}^- \in G^\infty_{\sigma,c}(\bar{x})$ denote the maximal walk and minimal walk, respectively.

(ii) If the setting satisfies (ES) *then for each $x \in \mathcal{A}$ we have $I_{\sigma,c}(x) = [0, \lambda(x)]$ and $I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]$.*

(iii) For each $x \in \mathcal{A}$ we have

$$I_{\sigma,c}(x) \cup (\lambda(x) + I_{\sigma,c}(\bar{x})) = [0, \lambda(x)].$$

If the setting does not satisfy (ES) *then this union is disjoint with respect to μ.*

(iv) For all $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ the set equation (3.1) *is disjoint with respect to μ.*

We need some, partly technical, lemmas to show the theorem. The first lemma refers to walks in the entire graph G_σ.

Lemma 3.2. *Let $x \in \mathcal{A}$ and $\mathfrak{x} \in G_\sigma^\infty(x)$. Then we have $0 \le \Lambda(\mathfrak{x}) \le \lambda(x)$. Analogously, for $\mathfrak{x} \in G_\sigma^\infty(\bar{x})$ we have $-\lambda(x) \le \Lambda(\mathfrak{x}) \le 0$.*

Proof. Let $\mathfrak{x} = (D_j.x_j)_{j\ge1} \in G_\sigma^\infty(x)$. By definition we have

$$\Lambda(\mathfrak{x}) = \sum_{j\ge1} \theta^{-j}\lambda(D_j).$$

Now we write $\Lambda(\mathfrak{x})$ as the limit of the partial sums, that is

$$\Lambda(\mathfrak{x}) = \lim_{n\to\infty}\sum_{j=1}^{n}\theta^{-j}\lambda(D_j) = \lim_{n\to\infty}\sum_{j=1}^{n}\theta^{-j}\langle \mathbf{l}(D_j),\mathbf{v}\rangle = \lim_{n\to\infty}\theta^{-n}\sum_{j=1}^{n}\theta^{n-j}\langle \mathbf{l}(D_j),\mathbf{v}\rangle .$$

Since $\mathbf{v}$ is a left eigenvector of $\mathbf{M}_\sigma$ with respect to the eigenvalue θ and by (2.1) we obtain for all $j \in \{1,\dots,n\}$

$$\begin{aligned}\theta^{n-j}\langle \mathbf{l}(D_j),\mathbf{v}\rangle &= \left\langle \mathbf{l}(D_j),\theta^{n-j}\mathbf{v}\right\rangle = \left\langle \mathbf{l}(D_j),\mathbf{v}\mathbf{M}_\sigma^{n-j}\right\rangle\\ &= \left\langle \mathbf{M}_\sigma^{n-j}\mathbf{l}(D_j),\mathbf{v}\right\rangle = \left\langle \mathbf{l}(\sigma^{n-j}(D_j)),\mathbf{v}\right\rangle .\end{aligned}$$

The additivity of the inner product thus yields

$$\Lambda(\mathfrak{x}) = \lim_{n\to\infty}\theta^{-n}\sum_{j=1}^{n}\left\langle \mathbf{l}(\sigma^{n-j}(D_j)),\mathbf{v}\right\rangle = \lim_{n\to\infty}\theta^{-n}\langle \mathbf{l}(X_n),\mathbf{v}\rangle = \lim_{n\to\infty}\theta^{-n}\lambda(X_n)$$

where

$$X_n = \sigma^{n-1}(D_1)\sigma^{n-2}(D_2)\cdots\sigma(D_{n-1})D_n.$$

Now observe that by Item 2 of Proposition 2.2 we have $(X_n,x_n) \in G^1_{\sigma^n}(x)$ and we conclude that $X_n \in \mathcal{A}^*$ is a prefix of $\sigma^n(x)$. Since λ is order-preserving we obtain

$$0 \le \Lambda(\mathfrak{x}) = \lim_{n\to\infty}\theta^{-n}\lambda(X_n) \le \lim_{n\to\infty}\theta^{-n}\lambda(\sigma^n(x)) = \lambda(x).$$

The analogue statement for a path that starts in $\bar{x}$ can be shown similarly. □

The function Λ does not act on G_σ^∞ in an order-preserving way, i.e. for each $x \in \mathcal{A}\cup\overline{\mathcal{A}}$ and two walks $\mathfrak{x},\mathfrak{x}' \in G_\sigma^\infty(x)$ with $\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}'$ we may not expect that $\Lambda(\mathfrak{x}) \le \Lambda(\mathfrak{x}')$ holds. The next result shows that Λ preserves the order on the subgraph $G_{\sigma,c}$ provided that the setting (σ,c) satisfies the ordering property (O).

Lemma 3.3. *Let $x \in \mathcal{A}\cup\overline{\mathcal{A}}$, $\mathfrak{x},\mathfrak{x}' \in G_{\sigma,c}^\infty(x)$, and suppose that (σ,c) satisfies* (O). *If $\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}'$ then $\Lambda(\mathfrak{x}) \le \Lambda(\mathfrak{x}')$.*

Proof. Let $\mathfrak{x} = (D_j,x_j)_{j\ge1}$, $\mathfrak{x}' = (D'_j,x'_j)_{j\ge1}$ and suppose that $\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}'$. We apply the idea used in the proof of Lemma 3.2, that is $\Lambda(\mathfrak{x}) = \lim_{n\to\infty}\theta^{-n}\lambda(X_n)$ and $\Lambda(\mathfrak{x}') = \lim_{n\to\infty}\theta^{-n}\lambda(X'_n)$, where

$$\begin{aligned}X_n &= \sigma^{n-1}(D_1)\sigma^{n-2}(D_2)\cdots\sigma(D_{n-1})D_n\\ X'_n &= \sigma^{n-1}(D'_1)\sigma^{n-2}(D'_2)\cdots\sigma(D'_{n-1})D'_n.\end{aligned}$$

Since (σ,c) satisfies (O) we immediately see that $X_n \preceq X'_n$ holds for all $n \ge 1$ and therefore $\Lambda(\mathfrak{x}) \le \Lambda'(\mathfrak{x})$. □

The proofs of the following results are based on the same notations that we carefully introduce here. Each time we consider an (arbitrary) letter $x \in \mathcal{A}$. For this letter we define for each $n \in \mathbb{N}$ the family of intervals $\mathfrak{J}_n = \{J_k^{(n)} : 0 \le k < |\sigma^n(x)|\}$ in the following way. Let $\sigma^n(x) = y_1^{(n)}y_2^{(n)}\dots y_{|\sigma^n(x)|}^{(n)}$. For each $k \in \{0,\dots,|\sigma^n(x)|\}$ we let $P_k^{(n)}$ denote the kth prefix of $\sigma^n(x)$, that is

$$\begin{aligned}P_0^{(n)} &= \varepsilon,\\ P_k^{(n)} &= y_1^{(n)}\cdots y_k^{(n)} \text{ for } k \ge 1.\end{aligned}$$

Now for each $k \in \{0,\dots,|\sigma^n(x)|-1\}$ we set

$$J_k^{(n)} := \left[\theta^{-n}\lambda\left(P_k^{(n)}\right),\theta^{-n}\lambda\left(P_{k+1}^{(n)}\right)\right].$$

Each of the intervals $J_k^{(n)}$ has length $\theta^{-n}\lambda\left(y_{k+1}^{(n)}\right) > 0$ and we obviously have

$$[0,\lambda(x)] = \bigcup_{k=0}^{|\sigma^n(x)|-1} J_k^{(n)},$$

where the union is disjoint with respect to the interior.

The main idea in the proofs below is the following observation. If $\mathfrak{x}$ is a walk that starts in x then for each $n \in \mathbb{N}$ the first n edges determine an index $k \in \{0,\dots,|\sigma^n(x)|-1\}$ such that $\Lambda(x) \in J_k^{(n)}$. Indeed, let $\mathfrak{x} = (D_j, x_j)_{j\geq 1}$. We clearly have

$$\Lambda(\mathfrak{x}) \in \sum_{j=1}^{n} \theta^{-j}\lambda(D_j) + \theta^{-n} I_{\sigma,c}(x_n)$$

and by using the techniques from the proof of Lemma 3.2 we see that

$$\Lambda(\mathfrak{x}) \in \theta^{-n}\lambda(D) + \theta^{-n} I_{\sigma,c}(x_n),$$

where $D = \sigma^{n-1}(D_1)\cdots\sigma(D_{n-1})D_n \in \mathcal{A}^*$ is a prefix of $\sigma^n(x)$ and $(D, x_n) \in G^1_{\sigma^n, c^{(n)}}(x)$.

We have to distinguish whether $x_n \in \mathcal{A}$ or $x_n \in \overline{\mathcal{A}}$. If $x_n \in \mathcal{A}$ then Dx_n is also a prefix of $\sigma^n(x)$ and by Lemma 3.2 we have

$$\Lambda(\mathfrak{x}) \in \theta^{-n}\lambda(D) + \theta^{-n} I_{\sigma,c}(x_n) \subset J_k^{(n)}$$

with $k = |D|$. If $x_n \in \overline{\mathcal{A}}$ then we similarly obtain that $\Lambda(\mathfrak{x}) \in J_k^{(n)}$ with $k = |D|-1$.

Lemma 3.4. *For each $x \in \mathcal{A}$ we have*

$$I_{\sigma,c}(x) \subseteq [0, \Lambda(\mathfrak{x}^+)],$$

where $\mathfrak{x}^+ \in G^\infty_{\sigma,c}(x)$ denotes the maximal walk, and

$$I_{\sigma,c}(\bar{x}) \subseteq [\Lambda(\mathfrak{x}^-), 0],$$

where $\mathfrak{x}^- \in G^\infty_{\sigma,c}(\bar{x})$ is the minimal walk.

Proof. We show the first statement explicitly. The second one can be proved analogously.

Let $x \in \mathcal{A}$, $\mathfrak{x}^+ = (D_j^+, x_j^+)_{j\geq 1} \in G^\infty_{\sigma,c}(x)$ be the maximal walk and $\mathfrak{x} = (D_j, x_j)_{j\geq1} \in G^\infty_{\sigma,c}(x)$ different from $\mathfrak{x}^+$. We have $\mathfrak{x} <_{\text{lex}} \mathfrak{x}^+$, thus there exists an index $n \geq 1$ such that $(D_j, x_j) = (D_j^+, x_j^+)$ holds for all $j < n$ and $(D_n, x_n) < (D_n^+, x_n^+)$. Then, by our considerations from above, $\mathfrak{x} \in J_k^{(n)}$ with $k \leq \left|\sigma^{n-1}(D_1)\right| + \cdots + |D_n|$ where the inequality is strict if $x_n \in \overline{\mathcal{A}}$. On the other hand $\mathfrak{x}^+ \in J_{k^+}^{(n)}$ with $k^+ = \left|\sigma^{n-1}(D_1)\right| + \cdots + |D_n|$. In the latter case we clearly have equality since $\mathfrak{x}^+$ is the maximal walk and, thus, $x_n^+ \in \mathcal{A}$. Since $(D_n, x_n) < (D_n^+, x_n^+)$ we either have $D_n < D_n^+$ - in this case $k < k^+$ obviously holds - or we have $D_n = D_n^+$. Then the inequality for k is strict and we have $k = k^+ - 1$. This immediately shows that $\Lambda(\mathfrak{x}) \leq \Lambda(\mathfrak{x}^+)$. □

Lemma 3.5. *Let $x \in \mathcal{A}$. If $I_{\sigma,c}(x) = [0, \lambda(x)]$ and $I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]$ then* (ES) *is satisfied.*

Proof. We only need the collection $\mathfrak{J}_1 = \left\{J_k^{(1)} : 0 \leq k < |\sigma(x)|\right\}$ here. Consider (3.1), i.e.

$$I_{\sigma,c}(x) = \bigcup_{(D_1,x_1)\in G^1_{\sigma,c}(x)} \theta^{-1}(\lambda(D_1) + I_{\sigma,c}(x_1)),$$

$$I_{\sigma,c}(\bar{x}) = \bigcup_{(D_1',x_1')\in G^1_{\sigma,c}(\bar{x})} \theta^{-1}(\lambda(D_1') + I_{\sigma,c}(x_1')).$$

For each edge $(D_1, x_1) \in G^1_{\sigma,c}(x)$ we have $\theta^{-1}(\lambda(D_1) + I_{\sigma,c}(x_1)) \subset J_k^{(1)}$ where $k = |D_1|$ if $x_1 \in \mathcal{A}$ or $k = |D_1| - 1$ if $x_1 \in \overline{\mathcal{A}}$. Since the interiors of the intervals in $\mathfrak{J}_1$ are disjoint we see that if $I_{\sigma,c}(x)$ and $\operatorname{int}\left(\mathrm{J}_{\mathrm{k}}^{(1)}\right)$ have non-empty intersection then $\{k, k+1\}$ and $c(x)$ have non-empty intersection. Therefore, the assumption $I_{\sigma,c}(x) = [0, \lambda(x)]$ implies that for each $k \in \{0,\dots,|\sigma(x)|-1\}$ either $k \in c(x)$ or $k+1 \in c(x)$ holds.

For $I_{\sigma,c}(\bar{x})$ we proceed similarly. Observe that

$$[-\lambda(x), 0] = \bigcup_{k=0}^{|\sigma(x)|} \left(-\lambda(x) + J_k^{(1)}\right)$$

and $\lambda(x) = \theta^{-1}\lambda(\sigma(x))$. For each edge $(D_1', x_1') \in G^1_{\sigma,c}(\bar{x})$ we have that $\overline{D_1'} \in \mathcal{A}^*$ is a suffix of $\sigma(x)$. Therefore

$$\lambda(x) + \theta^{-1}(\lambda(D_1') + I_{\sigma,c}(x_1')) = \theta^{-1}(\lambda(\sigma(x)D_1') + I_{\sigma,c}(x_1')) \subset J_k^{(1)}$$

where $k = |\sigma(x)| + |D_1'|$ if $x_1' \in \mathcal{A}$ or $k = |\sigma(x)| + |D_1'| - 1$ if $x_1' \in \overline{\mathcal{A}}$. We conclude that $I_{\sigma,c}(x) \cap \left(-\lambda(x) + \mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right)\right) \neq \varnothing$ implies that $\{k - |\sigma(x)|, k+1-|\sigma(x)|\} \cap c(\bar{x}) \neq \varnothing$. The assumption $I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]$ shows that for each $k \in \{0, \dots, |\sigma(x)| - 1\}$ either $k - |\sigma(x)| \in c(\bar{x})$ or $k + 1 - |\sigma(x)| \in c(\bar{x})$ holds.

Summing up, the two conditions on $c(x)$ and $c(\bar{x})$ and the fact that c is a coding prescription yield two necessary conditions. At first, $|\sigma(x)|$ is odd and $c(x)$ as well as $c(\bar{x})$ are subsets of $2\mathbb{Z}$. Secondly, for all letters y that appear in $\sigma(x)$ we have $I_{\sigma,c}(y) = [0, \lambda(y)]$ and $I_{\sigma,c}(\bar{x}) = [-\lambda(y), 0]$. By repeating the argumentation from above and by the primitivity of σ we finally see that (ES) must hold. □

Lemma 3.6. *If (σ, c) does not satisfy* (ES) *then for all $x \in \mathcal{A}$ the intersection $B(x) := I_{\sigma,c}(x) \cap (\lambda(x) + I_{\sigma,c}(\bar{x}))$ has zero Lebesgue measure.*

Proof. By (3.1) we have for each $x \in \mathcal{A}$ that

$$I_{\sigma,c}(x) = \bigcup_{(D,x_1) \in G^1_{\sigma,c}(x)} \theta^{-1}(\lambda(D) + I_{\sigma,c}(x_1)),$$

$$\lambda(x) + I_{\sigma,c}(\bar{x}) = \bigcup_{(D',x_1') \in G^1_{\sigma,c}(\bar{x})} \theta^{-1}(\lambda(\sigma(x)) + \lambda(D') + I_{\sigma,c}(x_1')).$$

We already observed that each term in both unions is contained in $J_k^{(1)}$ for a particular k and the intervals contained in the collection $\mathfrak{J}_1$ have pairwise disjoint interiors. This allows us to characterise $B(x)$ in terms of the collection $\{J_k^{(1)} \cap B(x) : 0 \le k < |\sigma(x)|\}$ - at least up to the finitely many boundary points $\left\{\theta^{-1}\lambda\left(P_k^{(1)}\right) : 0 \le k \le |\sigma(x)|\right\}$ that do not influence considerations concerning the Lebesgue measure μ.

Consider an arbitrary $k \in \{0, \dots, |\sigma(x)| - 1\}$. If $k \in c(x)$ as well as $k + 1 \in c(x)$ then neither $k - |\sigma(x)| \in c(\bar{x})$ nor $k + 1 - |\sigma(x)| \in c(\bar{x})$. In the proof of Lemma 3.5 we have seen that this implies that $\mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap (\lambda(\mathrm{x}) + \mathrm{I}_{\sigma,\mathrm{c}}(\bar{\mathrm{x}})) = \varnothing$ which shows that $B(x)$ does not contain any point of the (open) interval $\mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right)$. Similarly, if both $k - |\sigma(x)|$ and $k + 1 - |\sigma(x)|$ are contained in $c(\bar{x})$, then $\mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap \mathrm{B}(\mathrm{x}) = \varnothing$, too.

On the other hand, if $k \in c(x)$ and $k + 1 - |\sigma(x)| \in c(\bar{x})$ then

$$\mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap \mathrm{I}_{\sigma,\mathrm{c}}(\mathrm{x}) = \mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap \theta^{-1}\left(\lambda\left(\mathrm{P}_\mathrm{k}^{(1)}\right) + \mathrm{I}_{\sigma,\mathrm{c}}\left(\mathrm{y}_\mathrm{k}^{(1)}\right)\right),$$

$$\mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap (\lambda(\mathrm{x}) + \mathrm{I}_{\sigma,\mathrm{c}}(\bar{\mathrm{x}})) = \mathrm{int}\left(\mathrm{J}_\mathrm{k}^{(1)}\right) \cap \theta^{-1}\left(\lambda\left(\mathrm{P}_\mathrm{k}^{(1)}\right) + \lambda\left(\mathrm{y}_\mathrm{k}^{(1)}\right) + \mathrm{I}_{\sigma,\mathrm{c}}\left(\overline{\mathrm{y}_\mathrm{k}^{(1)}}\right)\right)$$

which yields that, up to the boundary points $\theta^{-1}\lambda\left(P_k^{(1)}\right)$ and $\theta^{-1}\lambda\left(P_{k+1}^{(1)}\right)$, we have $J_k^{(1)} \cap B(x) = \theta^{-1}\left(\lambda(P_k^{(1)}\right) + B\left(y_k^{(1)}\right)$. A similar observation can be made if $k + 1 \in c(x)$ and $k - |\sigma(x)| \in c(\bar{x})$.

Let

$$b(x) := \{k \in \{0, \dots, |\sigma(x)| - 1\} : k \in c(x), k + 1 - |\sigma(x)| \in c(\bar{x})\}$$
$$\cup \{k \in \{0, \dots, |\sigma(x)| - 1\} : k + 1 \in c(x), k - |\sigma(x)| \in c(\bar{x})\}.$$

By our considerations we conclude that

$$\mu(B(x)) = \sum_{k \in b(x)} \theta^{-1}\mu\left(B\left(y_k^{(1)}\right)\right).$$

For each $y \in \mathcal{A}$ let $b_{x,y} := \left\{k \in b(x) : y_k^{(1)} = y\right\}$. Then the equation from above rewrites as

$$\mu(B(x)) = \sum_{y=1}^{m} \theta^{-1} b_{x,y} \mu(B(y)).$$

We now represent this relation for all letters in terms of a matrix. In particular, let $\mathbf{B} = (b_{x,y})_{1 \le x,y \le m}$. Then

$$(\mu(B(1)), \ldots, \mu(B(m))) = (\mu(B(1)), \ldots, \mu(B(m))) \cdot \theta^{-1}\mathbf{B}.$$

The entries of the matrix $\mathbf{B}$ are non-negative integers and we obviously have that $\mathbf{B} \le \mathbf{M}_\sigma$ (the inequality holds component-wisely). Observe that equality holds if and only if (σ, c) satisfies (ES). Hence, the requirements on (σ, c) imply that $\mathbf{B} \ne \mathbf{M}_\sigma$. The matrix $\mathbf{M}_\sigma$ is primitive and we conclude from [43] that all eigenvalues of $\mathbf{B}$ are strictly smaller than θ in modulus, especially, the spectral radius of $\theta^{-1}\mathbf{B}$ is strictly smaller than 1 showing that $(\mu(B(1)), \cdots, \mu(B(m))) = \mathbf{0}$. □

Actually, the interested reader easily verifies that for each $x \in \mathcal{A}$ the set equation

$$B(x) = \bigcup_{\substack{(P,y,S) \in F_\sigma^1(x) \\ |P| \in b(x)}} \theta^{-1}(\lambda(P) + B(y))$$

even holds for the above excluded boundary points. This makes the collection $\{B(x) : x \in \mathcal{A}\}$ itself the invariant set list of a GIFS with uniform ratio θ^{-1} and construction matrix $\mathbf{B}$.

Lemma 3.7. *For all $x \in \mathcal{A}$ we have*

$$(\lambda(x) + I_{\sigma,c}(\bar{x})) \cup I_{\sigma,c}(x) = [0, \lambda(x)].$$

If (ES) *holds then $I_{\sigma,c}(x) = [0, \lambda(x)]$ and $I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]$.*

Proof. Let $x \in \mathcal{A}$ and $\xi \in [0, \lambda(x)]$. Again we consider the collections $\mathfrak{J}_n$. There exists a (not necessarily unique) sequence of intervals $\left(J_{k_n}^{(n)}\right)_{n \ge 1}$ such that $J_{k_n}^{(n)} \in \mathfrak{J}_n$ and $\xi \in J_{k_n}^{(n)}$ holds for all $n \ge 1$. Recall that $P_{k_n}^{(n)} \in \mathcal{A}^*$ is the prefix of $\sigma^n(x)$ with $\left|P_{k_n}^{(n)}\right| = k_n$ and $y_{k_n+1}^{(n)}$ is the $k_n + 1$st letter of $\sigma^n(x)$, hence

$$J_{k_n}^{(n)} = \theta^{-n}\left(\lambda\left(P_{k_n}^{(n)}\right) + \left[0, \lambda(y_{k_n+1}^{(n)})\right]\right) = \theta^{-n}\left(\lambda\left(P_{k_n}^{(n)} y_{k_n+1}^{(n)}\right) + \left[-\lambda\left(y_{k_n+1}^{(n)}\right), 0\right]\right).$$

We clearly have that

$$\xi = \lim_{n \ge \infty} \theta^{-n}\lambda(P_{k_n}^{(n)}) = \theta^{-n}\lambda(P_{k_n}^{(n)} y_{k_n+1}^{(n)}).$$

Now, for each $n \ge 1$ we either have $k_n \in c^{(n)}(x)$ or $k_n - |\sigma^n(x)| \in c^{(n)}(\bar{x})$. Suppose that $k_n \in c^{(n)}(x)$ holds for infinitely many n. Then

$$\theta^{-n}\lambda\left(P_{k_n}^{(n)}\right) \in \theta^{-n}\left(\lambda\left(P_{k_n}^{(n)}\right) + I_{\sigma,c}\left(y_{k_n+1}^{(n)}\right)\right) \subset I_{\sigma,c}(x)$$

holds for infinitely many indices and since $I_{\sigma,c}(x)$ is a compact set we see that $\xi \in I_{\sigma,c}(x)$. If $k_n - |\sigma^n(x)| \in c^{(n)}(\bar{x})$ holds for infinitely many n then from

$$\theta^{-n}\lambda\left(P_{k_n}^{(n)}\right) - \lambda(x) \in \theta^{-n}\left(\lambda\left(\overline{\sigma^n(x)} P_{k_n}^{(n)}\right) + I_{\sigma,c}\left(y_{k_n+1}^{(n)}\right)\right) \subset I_{\sigma,c}(\bar{x})$$

we deduce that $\xi - \lambda(x) \in I_{\sigma,c}(\bar{x})$. This shows that $\xi \in (\lambda(x) + I_{\sigma,c}(\bar{x})) \cup I_{\sigma,c}(x)$ which yields the first statement of the lemma.

Now suppose that (ES) holds. If for infinitely many n we have $k_n \in c^{(n)}(x)$ then we have seen that $\xi \in I_{\sigma,c}(x)$. By the structure of the setting we see that for these indices $k_n + 1 \notin c^{(n)}(x)$ and, hence $k_n + 1 - |\sigma^n(x)| \in c^{(n)}(\bar{x})$. This implies that

$$\theta^{-n}\lambda\left(P_{k_n}^{(n)} y_{k_n+1}^{(n)}\right) - \lambda(x) \in \theta^{-n}\left(\lambda\left(\overline{\sigma^n(x)} P_{k_n}^{(n)} y_{k_n+1}^{(n)}\right) + I_{\sigma,c}\left(\bar{y}_{k_n+1}^{(n)}\right)\right) \subset I_{\sigma,c}(\bar{x})$$

and therefore $\xi - \lambda(x) \in I_{\sigma,c}(\bar{x})$, too. Analogously, if for infinitely many indices we have $k_n - |\sigma^n(x)| \in c^{(n)}(\bar{x})$ then $k_n - 1 \in c^{(n)}(x)$ and we also have both $\xi - \lambda(x) \in I_{\sigma,c}(\bar{x})$ as well as $\xi \in I_{\sigma,c}(x)$. □

Proof of Theorem 3.1. Item (i) is shown in Lemma 3.2 and Lemma 3.4. The assertion of Item (ii) has been shown in Lemma 3.7. Item (iii) follows from Lemma 3.6 and Lemma 3.7. Finally, for showing Item (iv) we consider the collection $\mathfrak{J}_1$ and observe that for two distinct edges (D_1, x_1) and (D_1', x_1') (where we may assume that $(D_1, x_1) \prec (D_1', x_1')$) the sets $\theta^{-1}(\lambda(D_1) + I_{\sigma,c}(x_1))$ and $\theta^{-1}(\lambda(D_1') + I_{\sigma,c}(x_1'))$ are contained in the same subset $J_k^{(1)}$ if and only if $x_1 = \overline{x_1'} \in \mathcal{A}$ and $k + 1 = |D_1 x_1| = |D_1'|$. If (ES) is satisfied then $c(x)$ does not contain $|D_1|$ and $|D_1| + 1$, thus, (3.1) has disjoint interior. If (ES) is not satisfied then the intersection $\theta^{-1}(\lambda(D_1) + I_{\sigma,c}(x_1))$ and $\theta^{-1}(\lambda(D_1 x_1) + I_{\sigma,c}(\bar{x}_1))$ has zero Lebesgue measure by Lemma 3.6. □

The next lemma states that for each vertex $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ there are at most four distinct walks with the same Λ-image. We will need this result in the next section, however, we put it here since for the prove we use (for the last time) the collections $\mathfrak{J}_n$. Observe that in Example 6.1 we study a particular setting where this maximum occurs.

Lemma 3.8. *Let $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $\mathfrak{x}_1, \dots, \mathfrak{x}_5 \in G_{\sigma,c}^{\infty}(x)$ distinct walks. Then there exist indices $r_1, r_2 \in \{1, \dots, 5\}$ such that $\Lambda(\mathfrak{x}_{r_1}) \neq \Lambda(\mathfrak{x}_{r_2})$.*

Proof. Let $x \in \mathcal{A}$ (for $x \in \overline{\mathcal{A}}$ the proof runs analogously) and $\mathfrak{x}_r := \left(D_j^{(r)}, x_j^{(r)}\right)_{j \geq 1}$ for each $r \in \{1, \dots, 5\}$. Choose $n \in \mathbb{N}$ that large such that the 5 initial walks $\left(D_j^{(r)}, x_j^{(r)}\right)_{j=1}^{n}$, $r \in \{1, \dots, 5\}$ are pairwise different. For each $r \in \{1, \dots, 5\}$ we have $\Lambda(\mathfrak{x}_r) \in \theta^{-n} \lambda\left(D^{(r)}\right) + \theta^{-n} I_{\sigma,c}\left(x_n^{(r)}\right)$ with $D^{(r)} := \sigma^{n-1}\left(D_1^{(r)}\right) \cdots \sigma\left(D_{n-1}^{(r)}\right) D_n^{(r)}$ and $\left(D^{(r)}, x_n^{(r)}\right) \in G^1_{\sigma^n, c^{(n)}}\left(x^{(r)}\right)$. Therefore $\Lambda(\mathfrak{x}_r) \in J_{k_r}^{(n)}$ with $k_r = \left|D^{(r)}\right|$ if $x_n^{(r)} \in \mathcal{A}$ and $k_r = \left|D^{(r)}\right| - 1$ if $x_n^{(r)} \in \overline{\mathcal{A}}$. By the choice of n and Item (3) of Proposition 2.2 we conclude that the edges $\left(D^{(1)}, x^{(1)}\right), \dots, \left(D^{(5)}, x^{(5)}\right)$ are pairwise distinct and we see by the Pigeonhole principle that there must be at least two indices $r_1, r_2 \in \{1, \dots, 5\}$ such that $k_{r_1} - k_{r_2} \geq 2$ which implies that $\Lambda(\mathfrak{x}_{r_1}) > \Lambda(\mathfrak{x}_{r_2})$ since each interval $J_k^{(n)}$ has positive length. □

We already have seen that for the Even setting $\mathfrak{C}_{\sigma,c}$ is a collection of closed intervals. In the next theorems we show that the same holds for the Continuous setting.

Theorem 3.9. *Let the setting (σ, c) satisfy (CS). Then there exist non-positive real numbers $v_1^-, v_2^-, \dots, v_m^-$ and non-negative real numbers $v_1^+, v_2^+, \dots, v_m^+$ such that for all $x \in \mathcal{A}$ we have $v_x^+ - v_x^- = \lambda(x)$, $I_{\sigma,c}(x) = [0, v_x^+]$, and $I_{\sigma,c}(\bar{x}) = [v_x^-, 0]$.*

Proof. Let $x \in \mathcal{A}$ and set $v_x^- := \Lambda(\mathfrak{x}^-)$ and $v_x^+ := \Lambda(\mathfrak{x}^+)$, where $\mathfrak{x}^- = (D_j^-, x_j^-)_{j \geq 1} \in G_{\sigma,c}^{\infty}(\bar{x})$ and $\mathfrak{x}^+ = (D_j^+, x_j^+)_{j \geq 1} \in G_{\sigma,c}^{\infty}(x)$ are the minimal walk and the maximal walk, respectively. Since (CS) holds we have $|D_1^+| - |D_1^-| + 1 = |\sigma(x)|$, hence $(D_1^-, x_1^-) = (\overline{\sigma(x)} D_1^+ x_1^+, \overline{x_1^+})$. By successive application of the same argument we see that $(D_j^-, x_j^-) = (\overline{\sigma(x_{j-1}^+)} D_j^+ x_j^+, \overline{x_j^+})$ holds for all $j \geq 1$. Therefore,

$$\begin{aligned} v_x^- = \Lambda(\mathfrak{x}^-) &= \theta^{-1}\left(-\lambda(\sigma(x)) + \lambda(D_1^+) + \lambda(x_1^+)\right) + \sum_{j \geq 2} \theta^{-j}\left(\lambda(\sigma(x_{j-1}^+) + \lambda(D_j^+) + \lambda(x_j^+)\right) \\ &= -\lambda(\sigma(x)) + \sum_{j \geq 1} \theta^{-j} \lambda(D_j^+) = \Lambda(\mathfrak{x}^+) - \lambda(\sigma(x)) = v_x^+ - \lambda(\sigma(x)). \end{aligned}$$

By Lemma 3.4 we have $I_{\sigma,c}(x) \subseteq [0, v_x^+]$ and $I_{\sigma,c}(\bar{x}) \subseteq [v_x^-, 0]$ and due to Lemma 3.7 equality must hold. □

Denote by $\mathbf{A}_{\sigma,c}$ the adjacency matrix of the graph $G_{\sigma,c}$. We consider it as composition of four submatrices

$$\mathbf{A}_{\sigma,c} = \begin{pmatrix} \mathbf{M}_+ & \overline{\mathbf{M}}_+ \\ \mathbf{M}_- & \overline{\mathbf{M}}_- \end{pmatrix} \in \mathbb{Z}^{2m \times 2m},$$

where each of these submatrices is of dimension $m \times m$. In particular, we have

$$\mathbf{M}_+ := \left(\#\left\{(D,x_1) \in G^1_{\sigma,c}(y) : x_1 = x\right\}\right)_{1\le x,y\le m},$$
$$\mathbf{M}_- := \left(\#\left\{(D,x_1) \in G^1_{\sigma,c}(y) : x_1 = \bar{x}\right\}\right)_{1\le x,y\le m},$$
$$\overline{\mathbf{M}}_+ := \left(\#\left\{(D,x_1) \in G^1_{\sigma,c}(\bar{y}) : x_1 = x\right\}\right)_{1\le x,y\le m},$$
$$\overline{\mathbf{M}}_- := \left(\#\left\{(D,x_1) \in G^1_{\sigma,c}(\bar{y}) : x_1 = \bar{x}\right\}\right)_{1\le x,y\le m}.$$

By observing the definition of $G_{\sigma,c}$ one readily verifies that

$$\mathbf{M}_+ + \overline{\mathbf{M}}_+ = \mathbf{M}_\sigma = \mathbf{M}_- + \overline{\mathbf{M}}_-.$$

Let $\mathbf{u} \in \mathbb{R}^m$ denote a right eigenvector of $\mathbf{M}_\sigma$ with respect to θ. Then we immediately see that $\binom{\mathbf{u}}{\mathbf{u}}$ is a corresponding right eigenvector of $\mathbf{A}_{\sigma,c}$ with respect to θ.

Let

$$\mathbf{w} := \left(\mu\left(I_{\sigma,c}(1)\right), \mu\left(I_{\sigma,c}(2)\right), \dots, \mu\left(I_{\sigma,c}(m)\right), \mu\left(I_{\sigma,c}(\bar{1})\right), \mu\left(I_{\sigma,c}(\bar{2})\right), \dots, \mu\left(I_{\sigma,c}(\bar{m})\right)\right) \in \mathbb{R}^{2m}.$$

By Item (iv) of Theorem 3.1 the union (3.1) is disjoint with respect to μ and, hence $\mathbf{w}$ is a left eigenvector of $\mathbf{A}_{\sigma,c}$ with respect to θ. Especially, if (σ,c) satisfies (CS) we have $\mathbf{w} = (v_1^+, \dots, v_m^+, -v_1^-, \dots, -v_m^-)$ and if (σ,c) satisfies (ES) then $\mathbf{w} = (\mathbf{v}\ \mathbf{v})$.

Example (Accompanying example). We want to illustrate the results by our example. We start with the collection $\mathfrak{C}_{\sigma,c_2}$. The minimal element of $G^\infty_{\sigma,c_2}(\bar{1})$ is $(\bar{2},\bar{1})^\omega$. With the calculations that we already performed we immediately obtain $v_1^- = -\sqrt{2}/2$ and $v_1^+ = \theta + v_1^- = 1 + \sqrt{2}/2$. The minimal element of $G^\infty_{\sigma,c_2}(\bar{2})$ is $(\varepsilon,\bar{1})(\bar{2},\bar{1})^\omega$. This yields that $v_2^- = \theta^{-1}v_1^- = -1 + \sqrt{2}/2$ and $v_2^+ = \sqrt{2}/2$. We therefore obtain

$$I_{\sigma,c_2}(\bar{1}) = [-\sqrt{2}/2, 0], \qquad I_{\sigma,c_2}(1) = [0, 1+\sqrt{2}/2],$$
$$I_{\sigma,c_2}(\bar{2}) = [-1+\sqrt{2}/2, 0], \qquad I_{\sigma,c_2}(2) = [0, \sqrt{2}/2].$$

The sets are sketched at the top right position in Figure 3.

Concerning the other settings that satisfy (CS): one easily verifies that for (σ,c_1) we have $I_{\sigma,c_1}(\bar{1}) = [-\theta,0]$, $I_{\sigma,c_1}(\bar{2}) = [-1,0]$ while $I_{\sigma,c_1}(1)$ and $I_{\sigma,c_1}(2)$ consist of the origin only; similarly the situation for the setting (σ,c_3) where $I_{\sigma,c_1}(1) = [0,\theta]$, $I_{\sigma,c_1}(2) = [0,1]$ and $I_{\sigma,c_1}(1) = I_{\sigma,c_1}(2) = \{0\}$ (see the left hand side of Figure 3).

Finally, we consider the setting (σ,c_4). Here Item (ii) if Theorem 3.1 implies that $I_{\sigma,c_4}(\bar{1}) = [-\theta,0]$, $I_{\sigma,c_4}(1) = [0,\theta]$, $I_{\sigma,c_4}(\bar{2}) = [-1,0]$ and $I_{\sigma,c_4}(2) = [0,1]$. This result is visualised bottom right in Figure 3.

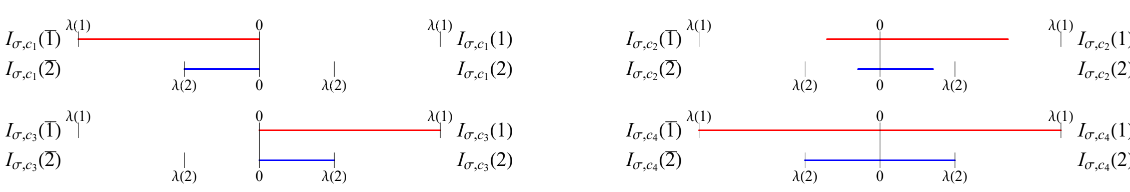

FIGURE 3. The shape of the elements of $\mathfrak{C}_{\sigma,c_1}$ (top left), $\mathfrak{C}_{\sigma,c_2}$ (top right), $\mathfrak{C}_{\sigma,c_3}$ (bottom left), $\mathfrak{C}_{\sigma,c_4}$ (bottom right).

We finish the section with some observations concerning higher powers of σ. By Proposition 2.2 a path of length $n \in \mathbb{N}$ on $G_{\sigma,c}$ corresponds to an edge of $G_{\sigma^n,c^{(n)}}$. Therefore, each walk on $G_{\sigma,c}$ corresponds to a walk on on $G_{\sigma^n,c^{(n)}}$. In particular, let $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $\mathfrak{x} = (D_j,x_j)_{j\ge1} \in G^\infty_{\sigma,c}(x)$. Then the corresponding walk on $G_{\sigma^n,c^{(n)}}$ is given by $\mathfrak{x}^{(n)} := (D'_j,x'_j)_{j\ge1} \in G^\infty_{\sigma^n c^{(n)}}(x)$ with $x'_j = x_{nj}$, $\quad D'_j = \sigma^{n-1}(D_{n(j-1)+1})\sigma^{n-2}(D_{n(j-1)+2})\cdots\sigma(D_{nj-1})D_{nj}$. Now observe that we have $\mathbf{M}_{\sigma^n} = \mathbf{M}_\sigma^n$. Therefore, if θ is the dominant root of $\mathbf{M}_\sigma$ then θ^n is the dominant root of $\mathbf{M}_{\sigma^n}$, and a left eigenvector $\mathbf{v}$ of $\mathbf{M}_\sigma$ with respect to θ is a left eigenvector of $\mathbf{M}_{\sigma^n}$ with respect to θ^n (and vice versa). We thus can define λ and subsequently Λ with respect to this eigenvector $\mathbf{v}$ for both σ as well as σ^n. By using the argumentation of Lemma 3.2 we obtain

$$\Lambda(\mathfrak{x}) = \sum_{j\geq 1} \theta^{-j}\lambda(D_j) = \sum_{j\geq 1} \theta^{-nj} \sum_{i=1}^{n} \theta^{n-i}\lambda(D_{n(j-1)+i}) = \sum_{j\geq 1} \theta^{-nj}\lambda(D'_j) = \Lambda(\mathfrak{x}').$$

We immediately see that for each n we have $\mathfrak{C}_{\sigma,c} = \mathfrak{C}_{\sigma^n,c^{(n)}}$. If we choose different eigenvectors for $\mathbf{M}_\sigma$ and $\mathbf{M}_{\sigma^n}$ then the sets contained in $\mathfrak{C}_{\sigma,c}$ and $\mathfrak{C}_{\sigma^n,c^{(n)}}$ coincide up to a multiplicative factor.

Example (Accompanying example). The latter considerations can be easily illustrated by our example. The dominant eigenvalue of $\mathbf{M}_{\sigma^2}$ is $\theta^2 = 3 + 2\sqrt{2}$ and $\mathbf{v} = (\theta, 1)$ is a corresponding left eigenvector. Then $\mathfrak{C}_{\sigma^2,c_1^{(2)}}$ coincides with $\mathfrak{C}_{\sigma,c_1}$.

For characterising $\mathfrak{C}_{\sigma^2,c'}$ we need additional calculations since the shape of this collection is not determined by a coding prescription with respect to σ. In particular, the interested reader verifies that $(\overline{21121}, \bar{1})^\omega$ is the minimal walk that starts in $\bar{1}$ while $(11, 2)^\omega$ is the maximal element of $G^\infty_{\sigma^2,c'}(2)$. Hence,

$$I_{\sigma^2,c'}(\bar{1}) = [-1/2 - \sqrt{2}, 0] \qquad I_{\sigma^2,c'}(1) = [0, 1/2]$$
$$I_{\sigma^2,c'}(\bar{2}) = \{0\} \qquad I_{\sigma^2,c'}(2) = [0, 1]$$

(see left hand side of Figure 4).

The sets contained in $\mathfrak{C}_{\sigma^2,c''}$ have a much more complex shape as it can be seen on the right hand side in Figure 4. We can calculate the Lebesgue measures by using the matrix

$$\mathbf{A}_{\sigma^2,c''} = \begin{pmatrix} 3 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

One easily verifies that $\mathbf{w} = (3/4, 5/4 - \sqrt{2}/2, 1/4 + \sqrt{2}, -1/4 + \sqrt{2}/2)$ is a left eigenvector of $\mathbf{A}$ with respect to the eigenvalue θ^2 such that the sum of the first and third entry equals $\lambda(1)$ and the sum of the other two entries yields $\lambda(2)$. We conclude that

$$\mu\left(I_{\sigma^2,c'}(\bar{1})\right) = \frac{1}{4} + \sqrt{2}, \quad \mu\left(I_{\sigma^2,c'}(1)\right) = \frac{3}{4}, \quad \mu\left(I_{\sigma^2,c'}(\bar{2})\right) = -\frac{1}{4} + \frac{\sqrt{2}}{2}, \quad \mu\left(I_{\sigma^2,c'}(2)\right) = \frac{5}{4} - \frac{\sqrt{2}}{2}.$$

FIGURE 4. On the left we see the shape of the elements of $\mathfrak{C}_{\sigma^2,c'}$. They are intervals since the setting (σ^2, c') satisfies (CS). On the right an approximation of the elements of $\mathfrak{C}_{\sigma^2,c''}$ is depicted. The setting (σ^2, c'') does neither satisfy (CS) nor (ES), hence the sets have a more complicated structure.

4. REPRESENTATION OF REAL NUMBERS

In this section we properly state and prove our main result, a generalisation of the Dumont-Thomas numeration. Let σ denote a primitive substitution over the alphabet $\mathcal{A}$ and c a coding prescription with respect to $\mathcal{A}$. We require (σ, c) to satisfy (CS) or (ES). Recall that in these cases $\mathfrak{C}_{\sigma,c}$ is a collection of closed intervals. More precisely, there exist non-negative real numbers $v_1^+, \ldots, v_m^+$, and non-positive real numbers $v_1^-, \ldots, v_m^-$ such that for each $x \in \mathcal{A}$ we have

$$I_{\sigma,c}(x) := [0, v_x^+], \qquad I_{\sigma,c}(\bar{x}) := [v_x^-, 0].$$

If (σ, c) fulfils (CS) then we have $v_x^+ - v_x^- = \lambda(x)$ (see Theorem 3.9) while for (σ, c) satisfying (ES) we have $v_x^+ = -v_x^- = \lambda(x)$ (see Theorem 3.1). We want to represent real numbers with respect to the dominant root θ of $\mathbf{M}_\sigma$. For obtaining uniqueness we have to restrict to the right-open intervals. We introduce the following notations: For each $x \in \mathcal{A}$ we define

$$\tilde{I}_{\sigma,c}(x) := [0, v_x^+), \qquad \tilde{I}_{\sigma,c}(\bar{x}) := [v_x^-, 0).$$

With this notation (3.1) can be rewritten as

$$\tilde{I}_{\sigma,c}(x) = \bigcup_{(D_1,x_1)\in G^1_{\sigma,c}(x)} \theta^{-1}\left(\lambda(D_1) + \tilde{I}_{\sigma,c}(x_1)\right) \tag{4.1}$$

and due to Item (iv) of Theorem 3.1 this union is completely disjoint. Note that all results in this section hold in an analogous way if we consider the left-open intervals.

Theorem 4.1. *Let* (σ, c) *satisfy* (CS) *or* (ES) *and* $x = x_0 \in \mathcal{A} \cup \overline{\mathcal{A}}$. *For each* $\gamma \in \tilde{I}_{\sigma,c}(x)$ *there exists a unique walk* $\mathfrak{x} = \mathfrak{x}(\gamma) = (D_j, x_j)_{j\geq 1} \in G^\infty_{\sigma,c}(x)$ *that satisfies*

$$\forall n \geq 0 : (D_j, x_j)_{j\geq n+1} \text{ is not the maximal element of } G^\infty_{\sigma,c}(x_n) \tag{4.2}$$

such that

$$\gamma = \Lambda(\mathfrak{x}) = \sum_{j\geq 1} \theta^{-j}\lambda(D_j). \tag{4.3}$$

If γ' *is another element of* $\tilde{I}_{\sigma,c}(x)$ *different from* γ *then we have*

$$\gamma < \gamma' \iff \mathfrak{x}(\gamma) \prec_{\mathrm{lex}} \mathfrak{x}(\gamma').$$

We call the representation (4.3) the (σ, c, x)-*expansion* of γ and $\mathfrak{d}_{\sigma,c,x}(\gamma) := (\lambda(D_j))_{j\geq 1}$ is the corresponding digit string. The specification of the initial vertex x is important since if γ is contained in $\tilde{I}_{\sigma,c}(x)$ as well as $\tilde{I}_{\sigma,c}(x')$ with $x \neq x'$ then the (σ, c, x)-expansion of γ does in general not coincide with the (σ, c, x')-expansion of γ, that is $\mathfrak{d}_{\sigma,c,x}(\gamma) \neq \mathfrak{d}_{\sigma,c,x'}(\gamma)$.

If $c = c_+$ then we immediately see that Theorem 4.1 holds since it corresponds to the Dumont-Thomas numeration stated in Theorem 2.6. Indeed, for each $x \in \overline{\mathcal{A}}$ the set $G^\infty_{\sigma,c_+}(x)$ consists of the zero-walk only that is simultaneously the maximal walk and, hence, forbidden. Therefore, the action takes place on the subgraph induced in G_{σ,c_+} by $\mathcal{A}$ and this subgraph corresponds to the prefix-suffix graph. Observe that for each $x \in \mathcal{A}$ the edge $(D, x_1) \in G^1_{\sigma,c_+}(x)$ is maximal if and only if $(D, x_1, \varepsilon) \in F^1_\sigma(x)$.

Proof of Theorem 4.1. We already observed that for $c = c_+$ the theorem corresponds to Theorem 2.6. Actually, our proof is inspired by the discourses on the Dumont-Thomas numeration in [9, 11]. We define

$$\mathfrak{A}_{\sigma,c} := \left\{(\xi, y) \in \mathbb{R} \times (\mathcal{A} \cup \overline{\mathcal{A}}) : \xi \in \tilde{I}_{\sigma,c}(y)\right\}.$$

Consider an element $(\xi, y) \in \mathfrak{A}_{\sigma,c}$. By the disjointness of the union (4.1) there exists a unique edge

$$e_{\sigma,c}(\xi, y) := (D, y') \in G^1_{\sigma,c}(y) \text{ such that } \xi \in \theta^{-1}\big(\lambda(D) + \tilde{I}_{\sigma,c}(y')\big),$$

especially $(\theta\xi - \lambda(D), y') \in \mathfrak{A}_{\sigma,c}$. We define the transformation $T_{\sigma,c} : \mathfrak{A}_{\sigma,c} \longrightarrow \mathfrak{A}_{\sigma,c}$ by

$$T_{\sigma,c} : (\xi, y) \longmapsto (\theta\xi - \lambda(D), y') \text{ where } (D, y') = e_{\sigma,c}(\xi, y).$$

Now let $\gamma \in \tilde{I}_{\sigma,c}(x)$ and observe that $(\gamma, x) \in \mathfrak{A}_{\sigma,c}$. We show that $\mathfrak{x} := (D_j, x_j)_{j\geq 1}$ with $(D_j, x_j) = e_{\sigma,c} \circ T^{j-1}_{\sigma,c}(\gamma, x)$ for each $j \geq 1$ is the unique walk $\mathfrak{x}(\gamma)$ as claimed in the statement of the theorem. Indeed, by construction $\mathfrak{x} \in G^\infty_{\sigma,c}(x)$ and for all $n \geq 1$ we have

$$\gamma \in \sum_{j=1}^{n} \theta^{-j}\lambda(D_j) + \theta^{-n}\tilde{I}_{\sigma,c}(x_n).$$

Since the intervals become arbitrary small we clearly have

$$\gamma = \sum_{j\geq 1} \theta^{-j}\lambda(D_j) = \Lambda(\mathfrak{x}).$$

Suppose that $\mathfrak{x}$ does not satisfy Condition (4.2). Then there exists an $n \in \mathbb{N}^0$ such that $\mathfrak{x}_n := (D_j, x_j)_{j\geq n+1}$ is the maximal walk starting in x_n and, hence $\Lambda(\mathfrak{x}_n)$ is the right endpoint of $I_{\sigma,c}(x_n)$. But by construction we have $(\Lambda(\mathfrak{x}_n), x_n) = T^n_{\sigma,c}(\gamma, x) \in \mathfrak{A}_{\sigma,c}$ and, thus, $\Lambda(\mathfrak{x}_n) \in \tilde{I}_{\sigma,c}(x_n)$, a contradiction.

Finally, we prove the uniqueness. Suppose that there were another walk $\mathfrak{x}' = (D'_j, x'_j)_{j\ge1} \in G^{\infty}_{\sigma,c}(x)$ different from $\mathfrak{x}$ that satisfies (4.2) with $\Lambda(\mathfrak{x}') = \gamma$. We show constructively that this yields a contradiction. We assume that $\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}'$ (for $\mathfrak{x}' \prec_{\text{lex}} \mathfrak{x}$ the proof runs analogously). Then there exists an index n' such that $(D_j, x_j) = (D'_j, x'_j)$ for $j < n'$ and $(D_{n'}, x_{n'}) = (D'_{n'}, x'_{n'})$. Since $\mathfrak{x}$ satisfies (4.2) we can find an index $n_1 > n'$ such that (D_{n_1}, x_{n_1}) is not the maximal element of $G^1_{\sigma,c}(x_{n_1-1})$. Let $\mathfrak{x}^{(1)} := \left(D^{(1)}_j, x^{(1)}_j\right)_{n\ge1} \in G^{\infty}_{\sigma,c}(x)$ such that $\left(D^{(1)}_j, x^{(1)}_j\right) = (D_j, x_j)$ for $j < n_1$ and $\left(D^{(1)}_j, x^{(1)}_j\right)_{n\ge n_1}$ is the maximal walk that starts in x_{n_1}. With the same argumentation we find another index $n_2 > n_1$ such that (D_{n_2}, x_{n_2}) is not the maximal edge and we let $\mathfrak{x}^{(2)}$ denote the walk whose first $n_2 - 1$ edges coincide with the fist $n_2 - 1$ edges of $\mathfrak{x}$ and from there on all edges are maximal. Finally, in the same way we construct $\mathfrak{x}^{(3)}$. Summing up, we have five distinct walks and by construction they fulfil

$$\mathfrak{x} \prec_{\text{lex}} \mathfrak{x}^{(3)} \prec_{\text{lex}} \mathfrak{x}^{(2)} \prec_{\text{lex}} \mathfrak{x}^{(1)} \prec_{\text{lex}} \mathfrak{x}'.$$

As $\Lambda(\mathfrak{x}) = \Lambda(\mathfrak{x}') = \gamma$ and (σ, c) satisfies (O) we conclude that

$$\Lambda(\mathfrak{x}) = \Lambda\left(\mathfrak{x}^{(3)}\right) = \Lambda\left(\mathfrak{x}^{(2)}\right) = \Lambda\left(\mathfrak{x}^{(1)}\right) = \Lambda(\mathfrak{x}') = \gamma$$

which contradicts Lemma 3.8. □

Let

$$\mathcal{N}_{\sigma,c} := \{\lambda(D) : (D, x') \in G^1_{\sigma,c}(x), x \in \mathcal{A} \cup \overline{\mathcal{A}}\}.$$

Then $\mathcal{N}_{\sigma,c}$ is the (finite) set of digits induced by the (σ, c, x)-expansions, independently of x. In other words, for each $x \in \mathcal{A} \cup \overline{\mathcal{A}}$ and $\gamma \in \tilde{I}_{\sigma,c}(x)$ the digit string $\mathfrak{d}_{\sigma,c,x}(\gamma)$ is an infinite sequence over $\mathcal{N}_{\sigma,c}$.

Observe that a vertex $x \in \mathcal{A}$ yields expansions of non-negative numbers while a vertex $x \in \overline{\mathcal{A}}$ provides expansions of negative numbers. This fact suggests to consider the graph $H_{\sigma,c}$ whose vertices correspond to pairs of letters (see Remark 2.3). We therefore state Theorem 4.1 in terms of $H_{\sigma,c}$ as a corollary.

Corollary 4.2. *Let* (σ, c) *satisfy* (CS) *or* (ES) *and* $ab = a_0b_0 \in \mathcal{A}^2$. *For each* $\gamma \in [v^-_a, v^+_b) = \tilde{I}_{\sigma,c}(\bar{a}) \cup \tilde{I}_{\sigma,c}(b)$ *there exists a unique walk* $\mathfrak{x} = (D_j, a_jb_j)_{j\ge1} \in H^{\infty}_{\sigma,c}(ab)$ *that satisfies*

$$\forall n \ge 0 : \mathfrak{x}_n := (D_j, a_jb_j)_{j\ge n+1} \text{ is not the maximal element of } H^{\infty}_{\sigma,c}(a_nb_n)$$

such that

$$\gamma = \Lambda(\mathfrak{x}) = \sum_{j\ge1} \theta^{-j} \lambda(D_j).$$

If γ' *is another element of* $[v^-_a, v^+_b)$ *different from* γ *then we have*

$$\gamma < \gamma' \iff \mathfrak{x}(\gamma) \prec_{\text{lex}} \mathfrak{x}(\gamma').$$

Proof. Observe that, if $(D, a_1b_1), (D', a'_1b'_1) \in H^{\infty}_{\sigma,c}(ab)$ are distinct edges, then we always have $D \prec D'$ or $D' \prec D$. We therefore can define $\prec$ on edges, paths and walks on $H_{\sigma,c}$ in a straightforward way. The rest of the proof is obvious. □

To better characterise the digit strings we introduce a special variant of the graph $G_{\sigma,c}$. In particular, define the *digit graph* $\hat{G}_{\sigma,c}$ to be the graph obtained from $G_{\sigma,c}$ by relabelling each edge (D, x') with $\lambda(D) \in \mathcal{N}_{\sigma,c}$. In accordance with our notations we let $\hat{G}^{\infty}_{\sigma,c}(x)$ denote the infinite walks that start in the vertex x. Observe that the digit graph $\hat{G}_{\sigma,c}$ is not right-resolving in the sense of [25] since in general the vertices have more than one outgoing edge with the same label. Here the larger graph $H_{\sigma,c}$ shows its advantage. Indeed, we can define the digit graph $\hat{H}_{\sigma,c}$ in an analogous way and this graph is right-resolving.

Example (Accompanying example). Each of the four settings (σ, c_1),(σ, c_2), (σ, c_3) and (σ, c_4) from our example satisfy the condition of Theorem 4.1.

At first observe that the setting (σ, c_3) corresponds to the Dumont-Thomas numeration. We clearly see by the respective graph in Figure 2 that the only walks that start in $\bar{1}$ or $\bar{2}$ are maximal walks and, hence, forbidden. For considering expansions it suffices to concentrate on the subgraph induced in $G_{\sigma,c}$ by the set $\mathcal{A}$ which corresponds to the prefix-suffix graph (up to the labels that do not contain the suffixes).

Since the setting (σ, c_3) does not yield new aspects we concentrate on the setting (σ, c_2). Here the digit set is given by $\mathcal{N}_{\sigma,c_2} := \{0, -1, 1+\sqrt{2}\}$. Let us consider the initial vertex $\bar{1}$. We can uniquely expand the elements of the interval $\tilde{I}_{\sigma,c_2}(\bar{1}) = [-\sqrt{2}/2, 0)$ with respect to the base θ. For instance, one easily verifies that

$$\mathfrak{x}(1-\sqrt{2}) = (\bar{2},2)(\varepsilon,1)^\omega \Longrightarrow \mathfrak{d}_{\sigma,c_2,\bar{1}}\left(1-\sqrt{2}\right) = -1,(0)^\omega,$$
$$\mathfrak{x}\left(1-2\sqrt{2}/7\right) = \left((\bar{2},2)(\varepsilon,1)(1,\bar{1})\right)^\omega \Longrightarrow \mathfrak{d}_{\sigma,c_2,\bar{1}}\left(1-2\sqrt{2}/7\right) = \left(-1,0,1+\sqrt{2}\right)^\omega.$$

Observe that $(\bar{2},\bar{1})\left((\varepsilon,\bar{2})(\varepsilon,\bar{1})\right)^\omega \in G^\infty_{\sigma,c_2}(\bar{1})$ does not satisfy (4.2).

With the initial vertex 2 we can represent the elements of $\tilde{I}_{\sigma,c_2}(2) = [0, \sqrt{2}/2)$. Some examples:

$$\mathfrak{x}\left(\sqrt{2}-1\right) = (\varepsilon,1)(1,1)(\varepsilon,1)(\varepsilon,11)^\omega \Longrightarrow \mathfrak{d}_{\sigma,c_2,2}\left(\sqrt{2}-1\right) = 0, 1+\sqrt{2}, (0)^\omega,$$
$$\mathfrak{x}\left(1-\sqrt{2}/2\right) = (\varepsilon,1)(1,\bar{1})(\bar{2},\bar{1})^\omega \Longrightarrow \mathfrak{d}_{\sigma,c_2,2}\left(1-\sqrt{2}/2\right) = 0, 1+\sqrt{2}, (-1)^\omega.$$

For obtaining the digit strings directly we replaced each label by the corresponding element of $\mathcal{N}_{\sigma,c_2}$. In this way we get the digit graph $\hat{G}_{\sigma,c_2}$ that is shown on the left hand side of Figure 5.

Due to Corollary 4.2 we can use the graph H_{σ,c_2} (depicted in the centre of Figure 5) to combine the expansions of negative numbers with respect to the vertex $\bar{1}$ and the expansions of non-negative numbers with respect to the vertex 2. Indeed, by starting in the vertex 12 we obtain the $(\sigma, c_2, \bar{1})$ expansions of the elements of $\tilde{I}_{\sigma,c_2}(\bar{1})$ and the $(\sigma, c_2, 2)$ expansions of the elements of $\tilde{I}_{\sigma,c_2}(2)$. By relabelling the edges in an analogous way we obtain the digit graph $\hat{H}_{\sigma,c_2}$ (right hand side of Figure 5) that also represents the digit strings, but this time in a right-resolving way.

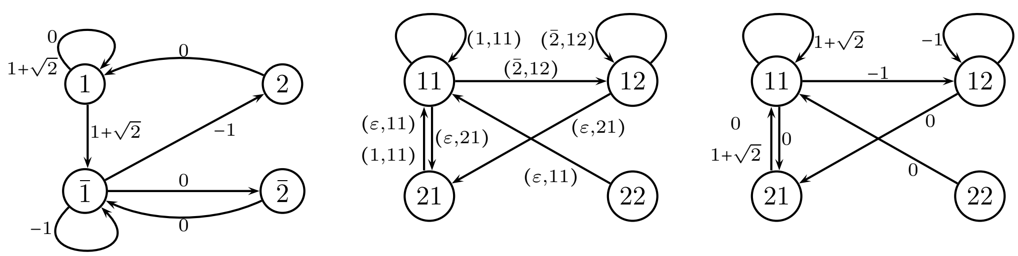

FIGURE 5. On the left we see the digit graph $\hat{G}_{\sigma,c_2}$ that represents the digit strings. It is clearly not right-resolving. In the centre we see the graph H_{σ,c_2}. By relabelling the edges in a proper way we obtain the digit graph $\hat{H}_{\sigma,c_2}$ on the right that is right-resolving.

The setting (σ^2, c') also satisfies the condition of Theorem 4.1. Here the representations are with respect to the base θ^2. The respective digit set is given by

$$\mathcal{N}_{\sigma^2,c'} = \left\{-5-3\sqrt{2}, -4-3\sqrt{2}, -3-2\sqrt{2}, -2-\sqrt{2}, -1-\sqrt{2}, 0, 1+\sqrt{2}, 2+2\sqrt{2}\right\}.$$

Remark 4.3. Theorem 4.1 allows us to represent substantially larger classes of real numbers. Let $x \in \mathcal{A}$ and suppose that $v_x^+ > 0$. Then for each non-negative $\gamma \in \mathbb{R}$ there exists an integer $n \geq 0$ such that $(\theta^{-n}\gamma, x) \in \mathfrak{A}_{\sigma,c}$. Let $(d_j)_{j\geq 1} = \mathfrak{d}_{\sigma,c,x}(\theta^{-n}\gamma)$. Then we obtain a representation of γ as

$$\gamma = \sum_{j\geq 1} d_j \theta^{n-j}.$$

Note that in order to maintain uniqueness we have to additionally require that n is chosen in a minimal way. A similar consideration for $\bar{x}$ allows us to represent arbitrary negative real numbers. Clearly, if $v_x^+ = 0$ or $v_x^- = 0$ this idea does not work for the entire real line but only for real numbers with an appropriate sign. In the rest of the article we will not further consider this approach and concentrate on the respective intervals $\tilde{I}_{\sigma,c}(x)$ and $\tilde{I}_{\sigma,c}(\bar{x})$.

Assume that for the left eigenvector $\mathbf{v}$ of $\mathbf{M}_\sigma$ with respect to the dominant eigenvalue θ we have $\mathbf{v} \in \mathbb{Q}(\theta)^m$. We say that the setting (σ, c) satisfies the periodicity property if

$$\text{(P)} \qquad \forall x \in \mathcal{A} \cup \overline{\mathcal{A}}, \gamma \in \mathbb{Q}(\theta) \cap \tilde{I}_{\sigma,c}(x) : \mathfrak{d}_{\sigma,c,x}(\gamma) \text{ is eventually periodic.}$$

A digit string $\mathfrak{d}_{\sigma,c,x}(\gamma)$ is finite if only finitely many digits are different from zero (i.e. the (σ, c, x)-expansion (4.3) is a finite sum). Motivated by the considerations concerning the Dumont-Thomas numeration (see Subsection 2.4) we say that the setting (σ, c) has the finiteness property (F) if

$$\text{(F)} \qquad \forall x \in \mathcal{A} \cup \overline{\mathcal{A}}, \gamma \in \mathbb{Z}[\mathbf{v}] \cap \tilde{I}_{\sigma,c}(x) : \mathfrak{d}_{\sigma,c,x}(\gamma) \text{ is finite.}$$

Obviously (σ, c_+) satisfies (F) if and only if the Dumont-Thomas numeration induced by σ satisfies the finiteness condition (as defined in Subsection 2.4). One easily obtains the following equivalent conditions: (σ, c) satisfies (P) if and only if for all $(\gamma, x) \in \mathfrak{A}_{\sigma,c}$ with $\gamma \in \mathbb{Q}(\theta)$ the orbit $\left\{T^n_{\sigma,c}(\gamma, x) : n \in \mathbb{N}\right\}$ is a finite set; (σ, c) satisfies (F) if and only if for all $(\gamma, x) \in \mathfrak{A}_{\sigma,c}$ with $\gamma \in \mathbb{Z}[\mathbf{v}]$ there exists an integer $n \geq 0$ such that $T^n_{\sigma,c}(\gamma, x) = (0, y)$. We start with a well-known classical result.

Proposition 4.4. *If θ is a Pisot number then (σ, c) satisfies* (P). *On the other hand, if (σ, c) satisfies* (P) *then θ is a Pisot number or a Salem number.*

Proof. This can be shown similarly as in [33] (see also [12]). □

Our intention in the present article is not to give characterisation results concerning periodicity and finiteness. Actually, we show that (P) does not depend on the choice of c. For (F) we show a similar result with a minor restriction, namely, that some special settings never satisfy (F).

Proposition 4.5. *Suppose that for a setting (σ, c) that satisfies* (O) *we have*

$$\exists x \in \mathcal{A} : I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]. \tag{4.4}$$

Then (σ, c) does not have the finiteness property (F).

Proof. We show indirectly that the $(\sigma, c, \bar{x})$-expansion of $\gamma = -\lambda(x) \in \mathbb{Z}[\mathbf{v}]$ is not finite. Let $\mathfrak{r}(\gamma) = (D_j, x_j)_{j \geq 1}$ and suppose that there exists an index n such that $D_j = \varepsilon$ for all $j > n$. By the considerations from the proof of Lemma 3.2 this implies that $\gamma = \theta^{-n}\lambda(X)$ with $X = \sigma^{n-1}(D_1) \cdots \sigma(D_{n-1}) D_n$. By Theorem 2.2 we have $(X, x_n) \in G^1_{\sigma^n, c^{(n)}}(\bar{x})$ which shows that $\sigma^n(\bar{x}) \prec X$, and since λ is order-preserving we obtain

$$\gamma = -\lambda(x) = \theta^{-n}\left(\sigma^n(\bar{x})\right) < \theta^{-n}\lambda(X) = \gamma,$$

a contradiction. □

Note that if (σ, c) satisfies (CS) then condition (4.4) means that $I_{\sigma,c}(x) = \{0\}$ and, hence $(0, x) \notin \mathfrak{A}_{\sigma,c}$. It is therefore not really astonishing that finiteness does not hold. But the proposition also induces that Even settings do not have the finiteness property.

Our main argument in context with periodicity and finiteness is that the transformation $T_{\sigma,c}$ is conjugate with the transformation T_{σ,c_+}.

Lemma 4.6. *Let (σ, c) satisfy* (CS) *or* (ES). *Then for all $(\xi, x) \in \mathfrak{A}_{\sigma,c}$ we have $S \circ T_{\sigma,c}(\xi, x) = T_{\sigma,c^+} \circ S(\xi, x)$ where*

$$S : (\xi, x) \longmapsto \begin{cases} (\xi, x) & \text{if } x \in \mathcal{A}, \\ (\xi - \lambda(x), \overline{x}) & \text{if } x \in \overline{\mathcal{A}}. \end{cases}$$

Proof. Let $(\xi, x) \in \mathfrak{A}_{\sigma,c}$ and $e_{\sigma,c}(\xi, x) = (D_1, y)$. Then $T_{\sigma,c}(\xi, x) = (\theta\xi - \lambda(D_1), y)$. We have four cases:

Case 1. $x \in \mathcal{A}, y \in \mathcal{A}$**:** Obviously

$$S \circ T_{\sigma,c}(\xi, x) = (\theta\xi - \lambda(D_1), y) \in \mathfrak{A}_{\sigma,c_+}.$$

On the other hand, since $y \in \mathcal{A}$ we have $D_1 \in \mathcal{A}^*$ and therefore $(D_1, y) \in G^1_{\sigma,c_+}(x)$. We immediately obtain

$$T_{\sigma,c_+} \circ S(\xi, x) = T_{\sigma,c_+}(\xi, x) = (\theta\xi - \lambda(D_1), y) = S \circ T_{\sigma,c}(\xi, x).$$

Case 2. $x \in \mathcal{A}, y \in \overline{\mathcal{A}}$**:** We clearly have

$$S \circ T_{\sigma,c}(\xi, x) = (\theta\xi - \lambda(D_1) - \lambda(y), \bar{y}) \in \mathfrak{A}_{\sigma,c_+}.$$

By the definition of G_σ we have $D_1 = D_1'\bar{y} \in \mathcal{A}^+$ and we conclude that $(D_1', \bar{y}) \in G^1_{\sigma,c_+}(x)$. This shows that

$$\begin{aligned} T_{\sigma,c_+} \circ S(\xi, x) = T_{\sigma,c_+}(\xi, x) =& (\theta\xi - \lambda(D_1'), \bar{y}) = \\ =& (\theta\xi - \lambda(D_1) - \lambda(y), \bar{y}) = S \circ T_{\sigma,c}(\xi, x) \end{aligned}$$

must hold.

Case 3. $x \in \overline{\mathcal{A}}, y \in \mathcal{A}$**:** Note that

$$S \circ T_{\sigma,c}(\xi, x) = (\theta\xi - \lambda(D_1), y) = (\theta(\xi + \lambda(\bar{x})) - \lambda(\sigma(\bar{x})D_1), y) \in \mathfrak{A}_{\sigma,c_+}.$$

From $D_1 \in \overline{\mathcal{A}}^+$ we see that $(\sigma(\bar{x})D_1, y) \in G^1_{\sigma,c_+}(\bar{x})$. We conclude that

$$T_{\sigma,c_+} \circ S(\xi, x) = T_{\sigma,c_+}(\xi + \lambda(\bar{x}), \bar{x}) = (\theta(\xi + \lambda(\bar{x})) - \lambda(\sigma(\bar{x})D_1), y) = S \circ T_{\sigma,c}(\xi, x).$$

Case 4. $x \in \overline{\mathcal{A}}, y \in \overline{\mathcal{A}}$**:** We have

$$\begin{aligned} S \circ T_{\sigma,c}(\xi, x) =& (\theta\xi - \lambda(D_1) - \lambda(y), \bar{y}) = (\theta\xi + \lambda(\sigma(\bar{x})) - \lambda(\sigma(\bar{x})) - \lambda(D_1 y), \bar{y}) \\ =& (\theta(\xi + \lambda(\bar{x})) - \lambda(\sigma(\bar{x})D_1 y), \bar{y}) \in \mathfrak{A}_{\sigma,c_+}. \end{aligned}$$

Since $D_1 \in \overline{\mathcal{A}}^*$ we obtain $(\sigma(\bar{x})D_1 y, \bar{y}) \in G^1_{\sigma,c_+}(\bar{x})$ and moreover

$$T_{\sigma,c_+} \circ S(\xi, y) = T_{\sigma,c_+}(\xi + \lambda(\bar{x}), \bar{x}) = (\theta(\xi + \lambda(\bar{x})) - (\sigma(\bar{x})D_1 y), \bar{y}) = S \circ T_{\sigma,c}(\xi, x).$$

□

Theorem 4.7. *Let* (σ, c) *be a setting that satisfies* (O). *Then* (σ, c) *has the periodicity property* (P) *if and only if* (σ, c_+) *has* (P).

If for all $x \in \mathcal{A}$ *we have* $-\lambda(x) \notin I_{\sigma,c}(\bar{x})$ *(i.e.* (σ, c) *does not satisfy* (4.4)*) then* (σ, c) *has the finiteness property* (F) *if and only if* (σ, c_+) *has* (F) *(which is equivalent to the Dumont-Thomas numeration induced by* σ *satisfying the finiteness condition).*

Proof. At first suppose that (σ, c_+) satisfies (P). Let $(\gamma, x) \in \mathfrak{A}_{\sigma,c}$ with $\gamma \in \mathbb{Q}(\beta)$ and set $(\gamma', x') := S(\gamma, x)$. We are interested in the orbit $O := \{T^n_{\sigma,c}(\gamma, x) : n \in \mathbb{N}\}$. We clearly have $\gamma' \in \mathbb{Q}(\beta)$ and since (σ, c_+) satisfies (P) the orbit $O' := \{T^n_{\sigma,c_+}(\gamma', x') : n \in \mathbb{N}\}$ is a finite set. Obviously $S(O) = O'$ by Lemma 4.6. If (σ, c) satisfies (CS) then S acts bijectively onto $\mathfrak{A}_{\sigma,c}$, hence $\#O = \#O'$. If (σ, c) satisfies (ES) then each $(\xi, y) \in \mathfrak{A}_{\sigma,c_+}$ has exactly two pre-images with respect to S, namely $(\xi, y) \in \mathfrak{A}_{\sigma,c}$ and $(\xi - \lambda(y), \bar{y}) \in \mathfrak{A}_{\sigma,c}$. We see that $\#O \leq 2\#O'$. Therefore, the orbit O is a finite set and (σ, c) satisfies (P).

Now assume that (σ, c_+) satisfies (F). Similarly as before we let $(\gamma, x) \in \mathfrak{A}_{\sigma,c}$ with $\gamma \in \mathbb{Z}[\mathbf{v}]$ and set $(\gamma', x') := S(\gamma, x)$. Then $\gamma' \in \mathbb{Z}[\mathbf{v}]$ and there exists an integer $n \geq 0$ such that

$$S \circ T^n_{\sigma,c}(\gamma, x) = T^n_{\sigma,c_+}(\gamma', x') = (0, y)$$

for an $y \in \mathcal{A}$ (where we used Lemma 4.6). By the requirements on (σ, c) we see that $(-\lambda(y), \bar{y}) \notin \mathfrak{A}_{\sigma,c}$ and conclude that $T^n_{\sigma,c}(\gamma, x) = (0, y)$.

The contrary statements can be shown analogously. □

We complete the section by a result that relates periodicity and finiteness properties of different powers of σ.

Proposition 4.8. *Let* (σ, c) *be a setting that satisfies* (O). *Then* (σ, c) *satisfies* (P) *(*(F)*, respectively), if and only if* $(\sigma^n, c^{(n)})$ *satisfies* (P) *(*(F)*, respectively).*

Proof. Trivial, since each walk on $G_{\sigma,c}$ corresponds to exactly one walk on $G_{\sigma^n,c^{(n)}}$. □

Example (Accompanying example). We consider the last time our example. From Proposition 4.5 we immediately deduce that the setting (σ, c_4) does not have the finiteness property (F). By Theorem 4.7 the Dumont-Thomas numeration induced by σ fulfils the finiteness condition if and only if (σ, c_3) satisfies (F) if and only if (σ, c_2) does which is in turn equivalent to (σ, c_1) having

(F). Observe that σ is a beta-substitution related with the beta-expansion with respect to the base θ, which is a Pisot number of degree 2. This implies that the Dumont-Thomas numeration induced by σ really satisfies the finiteness property. We will deal with beta-substitutions in the next section. Due to Proposition 4.8 we even have equivalence with the finiteness property of (σ^2, c').

5. Connections with generalised beta-expansions

It is well known that (classical) beta-expansions are intimately related with substitutions via the Dumont-Thomas numeration. This immediately implies that, for respective settings, Theorem 4.1 covers the beta-expansion. With our generalised approach we are able to show relations between generalised beta-expansions and substitutions.

Let $\delta \in [0,1)$ and $\beta > 1$. We consider transformations of the shape

$$T_{\beta,\delta} : [-\delta, 1-\delta) \longrightarrow [-\delta, 1-\delta), \xi \longmapsto \beta\xi - \lfloor \beta\xi + \delta \rfloor.$$

For each $\gamma \in [-\delta, 1-\delta)$ successive application of $T_{\beta,\delta}$ induces the (β,δ)-*expansion*

$$\gamma = d_1\beta^{-1} + d_2\beta^{-2} + d_3\beta^{-3} + \cdots \tag{5.1}$$

where $d_j = \beta T_{\beta,\delta}^{j-1}(\gamma) - T_{\beta,\delta}^{j}(\gamma) \in \mathbb{Z}$. We denote the digit string by $\mathfrak{d}_{\beta,\delta}(\gamma) := (d_j)_{j\geq 1}$. Observe that (5.1) is the unique radix representation of γ with respect to the base β and with integer digits that satisfies

$$\sum_{j\geq 1} d_{n+j}\beta^{-j} \in [-\delta, 1-\delta)$$

for all $n \geq 0$. An integers sequence $(d_j)_{j\geq 1}$ is called *admissible* (with respect to $T_{\beta,\delta}$) if there exists a $\gamma \in [-\delta, 1-\delta)$ such that $\mathfrak{d}_{\beta,\delta}(\gamma) = (d_j)_{j\geq 1}$. The (β,δ)-shift $\Omega_{\beta,\delta}$ is the symbolic dynamical system induced by the admissible sequences, that is

$$\Omega_{\beta,\delta} := \overline{\{\mathfrak{d}_{\beta,\delta}(\gamma) : \gamma \in [-\delta, 1-\delta)\}}.$$

The case $\delta = 0$ is the most famous one and corresponds to the classical beta-expansion introduced in [32]. For details we refer, *e.g.*, to the survey article [18]. For $\delta = 1/2$ we obtain the symmetric beta-expansion introduced in [6].

For a characterisation of the admissible sequences we follow the notations introduced in in [23]. Define the left-continuous counterpart of $T_{\beta,\delta}$ by

$$\hat{T}_{\beta,\delta} : (-\delta, 1-\delta] \longrightarrow (-\delta, 1-\delta], \xi \longmapsto \beta\xi + \lfloor -\beta\xi + 1 - \delta \rfloor$$

and for $\gamma \in (-\delta, 1-\delta]$ let $\mathfrak{d}^*_{\beta,\delta}(\gamma) := (d_j)_{j\geq 1}$ be the integer sequence satisfying $d_j = \beta\hat{T}_{\beta,\delta}^{j-1}(\gamma) - \hat{T}_{\beta,\delta}^{j}(\gamma)$. Observe that $\mathfrak{d}^*_{\beta,\delta}(\gamma)$ also provides a representation of γ since we have $\gamma = \sum_{j\geq 1}\beta^{-j}d_j$. For the characterisation of the admissible sequences the sequence $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ is important.

Proposition 5.1. *Let $\beta > 1$, $\delta \in [0,1)$. Then the following assertions hold.*

(1) An integer sequence $(d_j)_{j\geq 1}$ is admissible with respect to $T_{\delta,\beta}$ if and only if

$$\forall n \geq 1 : \mathfrak{d}_{\beta,\delta}(-\delta) \leq_{\text{lex}} (d_j)_{j\geq n} <_{\text{lex}} \mathfrak{d}^*_{\beta,\delta}(1-\delta).$$

(2) An integer sequence $(d_j)_{j\geq 1}$ is contained in $\Omega_{\beta,\delta}$ if and only if

$$\forall n \geq 1 : \mathfrak{d}_{\delta,\beta}(-\delta) \leq_{\text{lex}} (d_j)_{j\geq n} \leq_{\text{lex}} \mathfrak{d}^*_{\beta,\delta}(1-\delta).$$

(3) The (β,δ)-shift is sofic if and only if $\mathfrak{d}_{\beta,\delta}(-\delta)$ as well as $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ are eventually periodic sequences.

Proof. Item (1) follows immediately from [23, Theorem 2.5]. Item (2) is obvious (*cf.* Formula (6) in [23]). Finally, Item (3) is a consequence of [23, Proposition 2.14]. □

We want to remark that if both $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ are purely periodic then $\Omega_{\beta,\delta}$ is a shift of finite type. This can be easily shown by using the strategy from [6, Theorem 3.6].

In context with the classical case $\delta = 0$ the sequence $\mathfrak{d}^*_{\beta,0}(1)$ is frequently called the *characteristic sequence* and the results of the previous proposition can be found in [13, 30]. Observe that if this characteristic sequence is eventually periodic then β is said to be a *Parry-number* (in literature we also find the term *beta-number*). If the characteristic sequence is purely periodic then the term *simple Parry-number* has established. For more informations concerning Parry-numbers we refer to [14, 30, 38].

It has been observed in [17, 42] that for β a Parry number the structure of the admissible sequences with respect to $T_{\beta,0}$ can be described by a specific beta-substitution $\sigma_\beta = \sigma_{\beta,0}$. We follow [11] and state this relation in terms of the Dumont-Thomas numeration. Let $\mathfrak{d}^*_{\beta,0}(1) = e_1,\dots,e_q,(e_{q+1},\dots,e_{1+p})^\omega$. The beta-substitution $\sigma_{\beta,0}$ over the alphabet $\mathcal{A} = \{1,\dots,m = p+q\}$ is defined by

$$\sigma_{\beta,0}(x) = \begin{cases} 1^{e_x}(x+1) & \text{if } x \in \{1,\dots,m-1\}, \\ 1^{e_m}(q+1) & \text{if } x = m, \end{cases}$$

where 1^{e_x} denotes e_x repetitions of the letter 1 (and $1^0 = \varepsilon$).

Remark 5.2. Although we speak of *the* beta-substitution, $\sigma_{\beta,0}$ is actually not uniquely determined since we did not require the pre-period q and the period p to be chosen in a minimal way. However, the following results hold even when q and p are not chosen to be minimal (cf. Example 6.5).

Lemma 5.3 (*cf.* [11, 17, 42])**.** *Let $\beta > 1$ be a Parry-number and denote by $\sigma_{\beta,0}$ the corresponding beta-substitution. Then the dominant eigenvalue of $\mathbf{M}_{\sigma_{\beta,0}}$ is β.*

If we normalise the left eigenvector $\mathbf{v}$ of $\mathbf{M}_{\sigma_{\beta,0}}$ that corresponds to β such that we have $\lambda(1) = 1$ (i.e. the first entry of $\mathbf{v}$ equals 1) then for each $x \in \{1,\dots,m\}$ we have $\lambda(x) = \hat{T}^{x-1}_{\beta,0}(1) \le 1$.

Now we describe the exact relation between Dumont-Thomas numeration and beta-expansion.

Theorem 5.4 (*cf.* [11, 17, 42])**.** *Let $\beta > 1$ be a Parry-number, $\sigma_{\beta,0}$ the corresponding beta-substitution, and $\mathbf{v}$ a left eigenvector of $\mathbf{M}_{\sigma_{\beta,0}}$ with respect to β such that $\lambda(1) = 1$. Then for each $\gamma \in [0,1) = [0,\lambda(1))$ we have $\mathfrak{d}_{\sigma_{\beta,0},1}(\gamma) = \mathfrak{d}_\beta(\gamma)$, that is the $(\beta,0)$-expansion coincides with the $(\sigma_\beta,1)$-expansion.*

By observing Theorem 4.1 this immediately yields the following corollary.

Corollary 5.5. *Let $\beta > 1$ be a Parry-number, and σ_β, $\mathbf{v}$ as in Theorem 5.4. Then for each $\gamma \in [0,1) = \tilde{I}_{\sigma,c_+}(1)$ we have $\mathfrak{d}_{\sigma_\beta,c_+,1}(\gamma) = \mathfrak{d}_\beta(\gamma)$, i.e. the $(\sigma_\beta,c_+,1)$-expansion coincides with the $(\beta,0)$-expansion of γ.*

In Theorem 5.8 we amplify this result and show that Theorem 4.1 also covers beta-expansions for δ different from 0. We need some preliminary results. In the next lemma we fix a base β and relate the sequences $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ with the characteristic sequence $\mathfrak{d}^*_{\beta,0}(1)$.

Lemma 5.6. *Let $\beta > 1$, $\delta \in [0,1)$ and define $\mathfrak{d}_{\beta,\delta}(-\delta) = (l_j)_{j\ge1}$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta) = (r_j)_{j\ge1}$. If these sequences satisfy*

$$(\forall j \ge 1 : r_j \ge 0, l_j \le 0) \wedge (r_j - l_j \ne 0 \text{ for infinitely many indices}) \tag{5.2}$$

then

$$\mathfrak{d}^*_{\beta,0}(1) = \mathfrak{d}^*_{\beta,\delta}(1-\delta) - \mathfrak{d}_{\beta,\delta}(-\delta) = (r_j - l_j)_{j\ge1}.$$

Proof. Let $\mathfrak{d}^*_{\beta,0}(1) = (d_j)_{j\ge1}$. By definition have that $d_j = \beta\hat{T}^{j-1}_{\beta,0}(1) - \hat{T}^j_{\beta,0}(1)$ holds for all $j \ge 1$. Especially, d_j is the unique integer that satisfies

$$\beta\hat{T}^{j-1}_{\beta,0}(1) - d_j \in (0,1].$$

We show by induction on j that for each $j \in \mathbb{N}^0$ we have $\hat{T}^j_{\beta,\delta}(1-\delta) - T^j_{\beta,\delta}(-\delta) = \hat{T}^j_{\beta,0}(1)$. The base case $j = 0$ is clear. For the induction step assume that $\hat{T}^n_{\beta,\delta}(1-\delta) - T^n_{\beta,\delta}(-\delta) = \hat{T}^n_{\beta,0}(1)$ holds for some $n \in \mathbb{N}^0$. By definition we have

$$\begin{aligned} -\delta < \quad & \hat{T}^{n+1}_{\delta,\beta}(1-\delta) = \beta\hat{T}^n_{\delta,\beta}(1-\delta) - r_{n+1} && \le 1-\delta, \\ -\delta \le \quad & T^{n+1}_{\delta,\beta}(-\delta) \quad = \beta T^n_{\delta,\beta}(-\delta) - l_{n+1} && < 1-\delta. \end{aligned} \tag{5.3}$$

Since by Condition (5.2) $(l_j)_{j\ge1}$ is a sequence of non-positive integers we see that

$$T^{n+1}_{\beta,\delta}(-\delta) = \sum_{j\ge1} l_{j+n+1}\beta^{-j} \le 0.$$

Similarly, $\hat{T}^{n+1}_{\beta,\delta}(1-\delta) \geq 0$. Therefore, from (5.3) we obtain

$$\begin{array}{lll} 0 \leq & \hat{T}^{n+1}_{\beta,\delta}(1-\delta) = \beta\hat{T}^{n}_{\delta,\beta}(1-\delta) - r_{n+1} & \leq 1-\delta, \\ 0 \leq & -T^{n+1}_{\beta,\delta}(-\delta) = -\beta T^{n}_{\delta,\beta}(-\delta) + l_{n+1} & \leq \delta \end{array}$$

and addition yields

$$\begin{aligned} 0 \leq \hat{T}^{n+1}_{\beta,\delta}(1-\delta) - T^{n+1}_{\beta,\delta}(-\delta) &= \beta(\hat{T}^{n}_{\beta,\delta}(1-\delta) - T^{n}_{\beta,\delta}(-\delta)) - (r_{n+1} - l_{n+1}) \\ &= \beta\hat{T}^{n}_{\beta,0}(1) - (r_{n+1} - l_{n+1}) \leq 1. \end{aligned}$$

Finally, observe that the leftmost inequality is strict due to the additional requirement that $r_j - l_j \neq 0$ (and, hence, $r_j - l_j > 0$) for infinitely many indices. Since $r_{n+1} - l_{n+1}$ is an integer we conclude that $d_{n+1} = r_{n+1} - l_{n+1}$ and, hence $\hat{T}^{n+1}_{\beta,\delta}(1-\delta) - T^{n+1}_{\beta,\delta}(-\delta) = \hat{T}^{n}_{\beta,0}(1)$. □

Now consider a base $\beta > 0$ and an $\delta \in [0,1)$ such that (5.2) is satisfied. If $\mathfrak{d}_{\beta,\delta}(-\delta)$ as well as $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ are eventually periodic then, by Lemma 5.6, $\mathfrak{d}^*_{\beta,0}(1)$ is also eventually periodic. Hence, the soficness of $\Omega_{\beta,\delta}$ implies $\Omega_{\beta,0}$ to be sofic, too. Note that it is not clear whether a contrary statement also holds. Only for the purely periodic case we are able to show equivalence.

Lemma 5.7. *Let $\beta > 1$, $\delta \in [0,1)$ and suppose that $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ satisfy* (5.2)*. Then $\mathfrak{d}^*_{\beta,0}(1)$ is purely periodic with minimal period p if and only if $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ are purely periodic and the least common multiple of the minimal periods is p.*

Proof. Suppose that $\mathfrak{d}^*_{\beta,0}(1)$ is purely periodic with minimal period p, hence, $\hat{T}^{p}_{\beta,0}(1) = 1$. Now consider the proof of Lemma 5.6. We immediately see that in (5.3) equality must hold for the non-strict inequalities, that is $T^{p}_{\beta,\delta}(-\delta) = -\delta$ and $\hat{T}^{p}_{\beta,\delta}(1-\delta) = 1-\delta$. Therefore, $\mathfrak{d}_{\beta,\delta}(-\delta)$ as well as $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ are purely periodic with period p. Since $\hat{T}^{n}_{\beta}(1) \neq 1$ for $0 < n < p$ we also see that $T^{n}_{\beta,\delta}(-\delta) = -\delta$ and $\hat{T}^{n}_{\beta,\delta}(1-\delta) = 1-\delta$ cannot hold at the same time for $0 < n < p$, hence, the LCM of the minimal periods of $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ is at least p.

The other direction follows immediately from Lemma 5.6 □

We now show that in case of a sofic (β,δ)-shift such that (5.2) is satisfied there exists a particular (β,δ)-substitution $\sigma_{\beta,\delta}$ and a suitable coding prescription c_δ with respect to $\sigma_{\beta,\delta}$ such that we can retrieve the (β,δ)-expansions from Theorem 4.1.

Let $\mathfrak{d}_{\beta,\delta}(-\delta) = (l_j)_{j\geq 1}$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta) = (r_j)_{j\geq 1}$ be eventually periodic. We may assume that they have the same pre-period q and the same period p (possibly by considering common multiples), *i.e.*

$$\begin{aligned} \mathfrak{d}_{\beta,\delta}(-\delta) &= l_1, \dots, l_q, (l_{q+1}, \dots, l_{q+p})^\omega, \\ \mathfrak{d}^*_{\beta,\delta}(1-\delta) &= r_1, \dots, r_q, (r_{q+1}, \dots, r_{q+p})^\omega. \end{aligned}$$

In the purely periodic case we have $q = 0$. Now we define $\sigma_{\beta,\delta}$ over the alphabet $\mathcal{A} := \{1, \dots, m = p+q\}$ and the coding prescription c_δ with respect to $\sigma_{\beta,\delta}$ for each $x \in \mathcal{A}$ by

$$\begin{aligned} \sigma_{\beta,\delta} : x &\longmapsto \begin{cases} 1^{r_x}(x+1)1^{l_x} & \text{if } x \in \{1,\dots,m-1\}, \\ 1^{r_m}(q+1)1^{l_m} & \text{for } x = m, \end{cases} \\ c_\delta : x &\longmapsto \{0, \dots, r_x\}, \quad (\text{and therefore } c_\delta(\bar{x}) = \{-l_x, \dots, 0\}). \end{aligned} \tag{5.4}$$

Obviously the setting $(\sigma_{\beta,\delta}, c_\delta)$ satisfies (CS). Observe that due to Lemma 5.6 we know that $\mathfrak{d}_{\beta,\delta}$ corresponds, up to the order of the letters, with a beta-substitution $\sigma_{\beta,0}$. If $\mathfrak{d}_{\beta,\delta}(-\delta)$ or $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ is not purely periodic then this beta-substitution is possibly not the minimal one (*cf.* Remark 5.2 and Example 6.5). For this reason we wrote *a* beta-substitution. However, by Lemma 5.3 β is the dominant root of $M_{\sigma_{\beta,\delta}}$.

Theorem 5.8. *Let $\beta > 1$ be a real (algebraic) number, $\delta \in [0,1)$ such that $\Omega_{\beta,\delta}$ is sofic and* (5.2) *is satisfied. Define the setting $(\sigma_{\beta,\delta}, c_\delta)$ as in* (5.4) *and denote by* **v** *the left eigenvector* *of $M_{\sigma_{\beta,\delta}}$ with respect to the dominant root β such that $\lambda(1) = 1$. Then for each $x \in \mathcal{A}$ we have $I_{\sigma_{\beta,\delta},c_\delta}(x) = \left[0, \hat{T}^{x-1}_{\beta,\delta}(1-\delta)\right]$ and $I_{\sigma_{\beta,\delta},c_\delta}(\bar{x}) = \left[T^{x-1}_{\beta,\delta}(-\delta), 0\right]$. For each $(\gamma, x) \in \mathfrak{A}_{\sigma_{\beta,\delta},c_\delta}$ we have*

$$\mathfrak{d}_{\sigma_{\beta,\delta},c_\delta,x}(\gamma) = \mathfrak{d}_{\beta,\delta}(\gamma)$$

In other words, the $(\sigma_{\beta,\delta}, c_\delta, x)$-expansion coincides with the (β,δ)-expansion of γ.

Proof. Let $\mathfrak{d}_{\beta,\delta}(-\delta) = (l_j)_{j\geq 1}$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta) = (r_j)_{j\geq 1}$ By construction we have for each $x \in \mathcal{A}$ that $(1^{r_x}, y)$ is the maximal element of $G^1_{\sigma_{\beta,\delta},c_\delta}(x)$ and $(\bar{1}^{l_x}, \bar{y})$ is the minimal element of $G^1_{\sigma_{\beta,\delta},c_\delta}(\bar{x})$ where $y = x+1$ if $x \in \{1, \dots, m-1\}$ and $y = q+1$ if $x = m$. For each $x \in \mathcal{A}$ let $\mathfrak{x}^+_x$ and $\mathfrak{x}^-_{\bar{x}}$ denote the maximal element of $G^\infty_{\sigma_{\beta,\delta},c_\delta}(x)$ and the minimal element of $G^\infty_{\sigma_{\beta,\delta},c_\delta}(\bar{x})$, respectively. We immediately see that

$$\mathfrak{x}^+_x = \begin{cases} (1^{r_x}, x+1)\cdots(1^{r_q}, q+1)\big((1^{r_{q+1}}, q+2)\cdots(1^{r_{m-1}}, m)(1^{r_m}, q+1)\big)^\omega & \text{if } x \leq q, \\ \big((1^{r_x}, x+1)\cdots(1^{r_{m-1}}, m)(1^{r_m}, q+1)\cdots(1^{r_{x-1}}, x)\big)^\omega & \text{if } x \geq q+1, \end{cases}$$

and, by the choice of $\mathbf{v}$, $\Lambda(\mathfrak{x}^+_x) = \hat{T}^{x-1}_{\beta,\delta}(1-\delta)$. This shows that $I_{\sigma_{\beta,\delta},c_\delta}(x)$ has the stated shape. Analogously, we obtain that $\Lambda(\mathfrak{x}^-_{\bar{x}}) = T^{x-1}_{\beta,\delta}(-\delta)$.

Now let $(\gamma, x) \in \mathfrak{A}_{\sigma_{\beta,\delta},c_\delta}$ and $(\gamma', x') = T_{\sigma_{\beta,\delta},c_\delta}(\gamma, x)$. For finishing the proof we show that $\gamma' = T_{\beta,\delta}(\gamma)$. Indeed, we clearly have that $\gamma \in \tilde{I}_{\sigma_{\beta,\delta},c_\delta}(x) \subset [-\delta, 1-\delta)$, thus γ is contained in the domain of $T_{\beta,\delta}$. Now, $\gamma' = \beta\gamma - \lambda(D) \in \tilde{I}_{\sigma_{\beta,\delta},c_\delta}(x') \subset [-\delta, 1-\delta)$ with $(D, x') \in G^1_{\sigma_{\delta,\beta},c_\delta}(x)$. By construction we have $\lambda(D) \in \mathbb{Z}$, hence $\lambda(D) = \lfloor \beta\gamma + \delta \rfloor$. □

We say that the (β,δ)-expansion fulfils the finiteness condition if $\mathfrak{d}_{\beta,\delta}(\gamma)$ is finite for all $\gamma \in \mathbb{Z}[\beta^{-1}] \cap [-\delta, 1-\delta)$. There exists a large amount of research concerning this topic, especially for the classical case $\delta = 0$. In [19] it has been shown that the finiteness condition implies β to be a Pisot number but a precise characterisation of the Pisot numbers that induce the finiteness condition does not exist yet. For partly results we refer to [1, 7, 10, 21, 24].

The symmetric case $\delta = 1/2$ has been analysed in [6, 22]. For general $\delta \in [0,1)$ a research was performed in [39]. It turned out that for the case $\delta = 0$ a characterisation of bases β such that the $(\beta, 0)$-expansion fulfils the finiteness condition seems to be harder then for other choices of δ.

Following [11] the $(\beta,0)$-expansion fulfils the finiteness property if and only if the Dumont-Thomas numeration induced by the beta-substitution $\sigma_{\beta,0}$ fulfils the finiteness condition. The following corollary to Theorem 5.8 generalises this result.

Corollary 5.9. *Let $\beta > 1$ be a real (algebraic) number, $\delta \in [0,1)$ such that $\Omega_{\beta,\delta}$ is sofic and* (5.2) *is satisfied. Define the setting $(\sigma_{\beta,\delta}, c_\delta)$ as in* (5.4). *Then $(\sigma_{\beta,\delta}, c_\delta)$ satisfies* (F) *if and only if the (β,δ)-expansion fulfils the finiteness condition.*

Proof. Let $\gamma \in \mathbb{Z}[\beta^{-1}] \cap [-\delta, 1-\delta)$ and note that there exists an $n \in \mathbb{N}$ such that $T^n_{\beta,\delta}(\gamma) \in \mathbb{Z}[\beta]$. Thus it suffices to concentrate on the set $\mathbb{Z}[\beta]$. Now observe that by the choice of $\mathbf{v}$ we have $\lambda(1) = 1$. By Lemma 5.3 for $x = 1, \dots, m-1$ we have $\lambda(x+1) = \hat{T}^x_{\beta,0}(1) = \beta\lambda(x) - d_x$ with $d_x \in \mathbb{Z}$ where m is greater than or equal to the algebraic degree of β. From this we immediately see that $\mathbb{Z}[\mathbf{v}] = \mathbb{Z}[\beta]$ and therefore Theorem 5.8 shows the equivalence of the two notions of finiteness. □

We have seen that our theory unifies several notions of generalised beta-expansions and the respective notions of finiteness. However, there are still gaps. On one hand we had restrictions on the base β (the sequences $\mathfrak{d}_{\beta,\delta}(-\delta)$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta)$ have to satisfy (5.2)). On the other hand, there are further particular concepts of non-integer systems that we have not even discussed. As conclusion we will therefore state the following questions.

Open question 1. Let $\beta > 1$, $\delta \in [0,1)$ such that $\Omega_{\beta,\delta}$ is sofic and suppose that (5.2) is not satisfied. Does there exist a setting (σ, c) such that we can recover the (β,δ)-expansion by Theorem 4.1.

Open question 2. Does Theorem 4.1 cover further known systems of numeration.

6. Examples

Example 6.1. Let $\sigma : 1 \mapsto 121, 2 \mapsto 12$ over the alphabet $\mathcal{A} = \{1,2\}$. The dominant root of $\mathbf{M}_\sigma$ is $\theta := {}^{(3+\sqrt{5})}/_2$, the square of the golden mean. We consider the coding prescription c determined by $c(1) = \{0\}$ and $c(2) = \{0,1\}$. The setting (σ, c) clearly satisfies (CS). The graph $G_{\sigma,c}$ is depicted in Figure 6 on the left hand side. We see that it has three strongly connected components. In fact, it is straightforward that $I_{\sigma,c}(1) = I_{\sigma,c}(\bar{2}) = \{0\}$ which implies that $I_{\sigma,c}(\bar{1}) = [-\lambda(1), 0]$ and $I_{\sigma,c}(2) = [0, \lambda(2)]$.

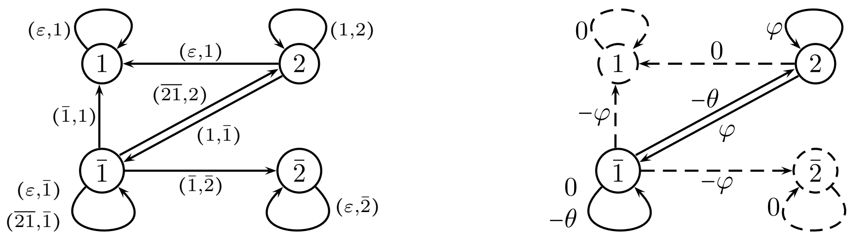

Figure 6. Left: The graph $G_{\sigma,c}$ for the setting (σ, c) in Example 6.1. It has three strongly connected components. Right: The digit graph $\hat{G}_{\sigma,c}$. We can reduce it by the vertices 1 and $\bar{2}$ since the walks that start in these vertices do not satisfy (4.2)

.

Let $\mathbf{v} = (\varphi, 1)$, where $\varphi := \sqrt{\theta} = \theta - 1 = {}^{(1+\sqrt{5})}/_2$ is the golden mean. One easily verifies that $\mathbf{v}$ is a left eigenvector of $\mathbf{M}_\sigma$ with respect to θ. At first we observe that for this setting the upper bound of Lemma 3.8 is really achieved. For example, consider the vertex $\bar{1}$ and the walks

$$\mathfrak{x}^{(1)} = (\bar{1},\bar{2})(\varepsilon,\bar{2})^\omega \in G^\infty_{\sigma,c}(\bar{1}), \qquad \mathfrak{x}^{(2)} = (\bar{1},1)(\varepsilon,1)^\omega \in G^\infty_{\sigma,c}(\bar{1}),$$
$$\mathfrak{x}^{(3)} = (\overline{21},2)(1,2)^\omega \in G^\infty_{\sigma,c}(\bar{1}), \qquad \mathfrak{x}^{(4)} = (\varepsilon,\bar{1})(\overline{21},\bar{1})^\omega \in G^\infty_{\sigma,c}(\bar{1}).$$

Then we have $\Lambda(\mathfrak{x}^{(1)}) = \Lambda(\mathfrak{x}^{(2)}) = \Lambda(\mathfrak{x}^{(3)}) = \Lambda(\mathfrak{x}^{(4)}) = -\varphi^{-1}$.

Our setting suggests only one way of representation: $(\sigma, c, 2)$-expansions for non-negative numbers and $(\sigma, c, \bar{1})$-expansions for negative numbers. The vertices 1 and $\bar{2}$ are not involved. The respective digit graph $\hat{G}_{\sigma,c}$ is shown in Figure 6 on the right. The (unique) $(\sigma, c, \bar{1})$-expansion of $-\varphi^{-1}$ is given by $\mathfrak{x}^{(4)}$ (all other walks do not satisfy (4.2)). The setting (σ, c) does not have (F) by Proposition 4.5.

We want to remark that our expansions (with respect to the base $\theta = \varphi^2$) can be easily seen as expansions with respect to the base φ. Each digit in the digit string corresponds to two consecutive digits in the new expansion. In this way we actually obtain integer digits.

Example 6.2. Let β be the dominant root of the polynomial $t^3 - 2t^2 - 1$. Then β is a Pisot number and $\mathfrak{d}^*_{\beta,1/2}(1/2) = -\mathfrak{d}_{\beta,1/2}(-1/2) = (1,0,0)^\omega$, hence, the conditions of Theorem 5.8 are satisfied. We define the setting $(\sigma_{\beta,1/2}, c_{1/2})$ as in (5.4) by $\sigma_{\beta,1/2} : 1 \mapsto 121, 2 \mapsto 3, 3 \mapsto 1$ and $c_{1/2} : 1 \mapsto \{0,1\}, 2 \mapsto \{0\}, 3 \mapsto \{0\}$. The graph $G_{\sigma_{\beta,1/2},c_{1/2}}$ is depicted in Figure 7. The vector $\mathbf{v} = (1, \beta - 2, \beta^2 - 2\beta)$ is a left eigenvector of $\mathbf{M}_{\sigma_{\beta,1/2}}$ with respect to the eigenvalue β. By Theorem 5.8 we have $I_{\sigma_{\beta,1/2}}(\bar{1}) = [-1/2, 0]$ and $I_{\sigma_{\beta,1/2}}(1) = [0, 1/2]$. For each $\gamma \in [-1/2, 1/2)$ we retrieve the symmetric beta-expansion of γ by Theorem 4.1: the $(\sigma_{\beta,1/2}, c_{1/2}, \bar{1})$-expansion if $\gamma \in [-1/2, 0)$ and the $(\sigma_{\beta,1/2}, c_{1/2}, 1)$-expansion if $\gamma \in [0, -1/2)$.

Corollary 4.2 provides a way to obtain the $(\beta, 1/2)$-expansions as walks that start in the unique initial vertex 11. Consider the digit graph $\hat{H}_{\sigma_{\beta,1/2},c_{1/2}}$ (see Figure 8). We can remove the dashed vertices 22, 23, 32, 33 since they have no incoming edge. We see that the remaining graph corresponds to the minimal right resolving presentation graph of the $(\beta, 1/2)$-shift.

Observe that the $(\beta, 1/2)$-expansions fulfil the finiteness condition. Therefore, by Corollary 5.9, $(\sigma_{\beta,1/2}, c_{1/2})$ has (F), by Theorem 4.7, $(\sigma_{\beta,1/2}, c_+)$ has (F), which shows that the Dumont-Thomas numeration induced by $\sigma_{\beta,1/2}$ also fulfils the finiteness condition.

FIGURE 7. On the left we see the graph $G_{\sigma_{\beta,1/2},c_{1/2}}$ for the setting $\sigma_{\beta,1/2}, c_{1/2}$ discussed in Example 6.2. It corresponds to the symmetric beta-expansion with respect to the dominant root of $t^3 - 2t^2 - 1$. On the right we see the digit graph $\hat{G}_{\sigma_{\beta,1/2},c_{1/2}}$.

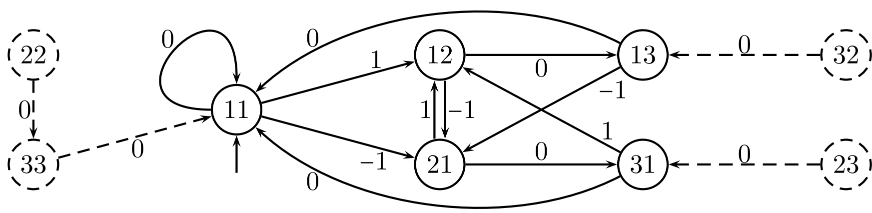

FIGURE 8. By considering the digit graph $\hat{H}_{\sigma_{\beta,1/2},c_{1/2}}$ we can obtain all $(\beta, 1/2)$-expansions as walks that start in the unique initial vertex 11. We can omit the dashed vertices since they have no incoming edge.

Note that in the next example we show that the substitution $\tilde{\sigma}_\beta$ provides another possibility to expand real numbers with respect to β by using integer digits.

Example 6.3. We consider the substitution $\sigma : 1 \mapsto 121, 2 \mapsto 3, 3 \mapsto 1$ over the alphabet $\mathcal{A}$ (*cf.* Example 6.2). Its dominant eigenvalue θ is the real root of $t^3 - 2t^2 - 1$. Let $\mathbf{v} := (\theta^2/\theta^2+1, 1/\theta^2+1, \theta/\theta^2+1)$ and observe that it is a left eigenvector of $\mathbf{M}_\sigma$ with respect to θ. In this example we want to discuss expansions with respect to a setting that satisfies (ES). Thus, we consider the coding prescription c defined by $c(1) = \{0, 2\}$, $c(2) = c(3) = \{0\}$. The respective graph $G_{\sigma,c}$ is depicted in Figure 9 (left).

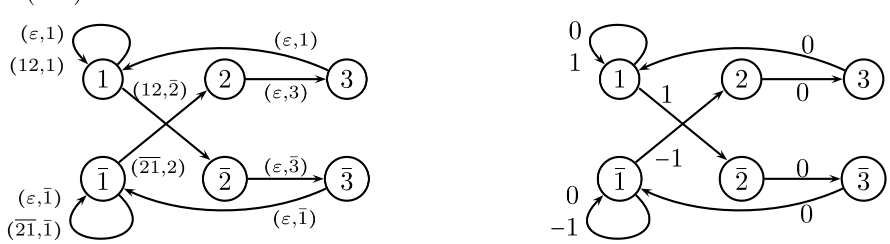

FIGURE 9. On the left the graph $G_{\sigma,c}$ for the Even setting (σ, c) studied in Example 6.3 is depicted. On the right we see the digit graph $\hat{G}_{\sigma,c}$.

We are interested in the induced expansions. By Theorem 3.1 we have for all $x \in \{1, 2, 3\}$ that $I_{\sigma,c}(\bar{x}) = [-\lambda(x), 0]$ and $I_{\sigma,c}(x) = [0, \lambda(x)]$. Note that $\lambda(1) > \lambda(2) > \lambda(3)$, hence we concentrate on the vertices 1 and $\bar{1}$. In this way we can uniquely expand the real numbers in $[-(\theta-1)^{-1}, (\theta-1)^{-1})$ with respect to the base θ. The right hand side of Figure 9 shows the digit graph $\hat{G}_{\sigma,c}$ We see that (due to our choice of $\mathbf{v}$) the digit set is $\mathcal{N}_{\sigma,c} = \{-1, 0, 1\}$. In the induced digit strings the digits 1 and -1 appear only with at least two 0s in between. From Proposition 4.5 we deduce that the setting (σ, c) does not satisfy the finiteness property (F).

Example 6.4. Let $\beta := (3+\sqrt{17})/2$, a Pisot number which is the positive root of $t^2 - 3t - 2$. The $(\beta, 0)$-shift $\Omega_{\beta,0}$ is of finite type since the characteristic sequence is given by $\mathfrak{d}^*_{\beta,0}(1) = (3, 1)^\omega$. The corresponding substitution is $\sigma_{\beta,0} : 1 \mapsto 1112, 2 \mapsto 11$ over the alphabet $\mathcal{A} = \{1, 2\}$.

Now let $\delta := (\beta - 1)^{-1}$. Then $\mathfrak{d}_{\beta,\delta}(-\delta) = (-1)^\omega$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta) = (2,0)^\omega$, thus Condition (5.2) holds and the subshift $\Omega_{\beta,\delta}$ is sofic (*cf.* Lemma 5.7) (in fact, it is also a shift of finite type). We follow (5.4) and define the setting $(\sigma_{\beta,\delta}, c_\delta)$ by

$$\begin{aligned} \sigma_{\beta,\delta} &: 1 \mapsto 1121, 2 \mapsto 11, \\ c_\delta &: 1 \mapsto \{0,1,2\}, 2 \mapsto \{0\}. \end{aligned}$$

The graph $G_{\sigma_{\beta,\delta},c_\delta}$ is depicted on the left in Figure 10. Fix the left eigenvector $\mathbf{v} := (1, \beta - 3)$. Then, due to Theorem 5.8, for each $\gamma \in [-\delta, 1-\delta)$ we can retrieve the (β,δ)-expansion by the $(\sigma_{\beta,\delta}, c_\delta, \bar{1})$-expansion if $\gamma < 0$ and by the $(\sigma_{\beta,\delta}, c_\delta, 1)$-expansion if $\gamma \geq 0$. The right hand side of Figure 10 shows the digit graph $\hat{G}_{\sigma_{\beta,\delta},c_\delta}$.

Observe that in this example $\sigma_{\beta,0}$ and $\sigma_{\beta,\delta}$ are conjugate substitutions which means that there exists a word $X \in \mathcal{A}^* \cup \overline{\mathcal{A}}^*$ such that for all $x \in \mathcal{A}$ we have $\sigma_{\beta,\delta}(x) = X\sigma_{\beta,0}(x)\overline{X}$ (modulo $\sim$).

FIGURE 10. Left: The graph $G_{\sigma_{\beta,\delta},c_\delta}$ for the setting $(\sigma_{\beta,\delta}, c_\delta)$ in Example 6.4. The setting is related with the (β,δ)-expansions for $\beta = (3+\sqrt{17})/2$ and $\delta = (\beta-1)^{-1}$. Right: The corresponding digit graph $\hat{G}_{\sigma_{\beta,\delta},c_\delta}$. The (β,δ)-expansions correspond to the walks that start in 1 and $\bar{1}$.

Example 6.5. We let $\beta := (5+\sqrt{21})/2$. As in the previous examples β is a Pisot number and its minimal polynomial equals $t^2 - 5t + 1$. One easily verifies that $\mathfrak{d}^*_{\beta,0}(1) = 4,(3)^\omega$, thus, the corresponding $(\beta,0)$-shift $\Omega_{\beta,0}$ is sofic but not of finite type. Both the minimal pre-period and the minimal period of $\mathfrak{d}^*_{\beta,0}(1)$ are 1. The corresponding beta-substitution is given by $\sigma_{\beta,0} : 1 \mapsto 11112, 2 \mapsto 1112$ over the alphabet $\mathcal{A} = \{1,2\}$. As observed in Remark 5.2 the beta-substitution $\sigma_{\beta,0}$ is not uniquely determined. For example, Lemma 5.3 and Theorem 5.4 also hold for the substitution $\sigma'_{\beta,0} : 1 \mapsto 11112, 2 \mapsto 1113, 2 \mapsto 1112$ over the alphabet $\mathcal{A} = \{1,2,3\}$. Note that $\mathbf{M}_{\sigma'_{\beta,0}}$ possesses the additional eigenvalue -1 (hence, the minimal polynomial of β is a proper factor of the characteristic polynomial of $\mathbf{M}_{\sigma'_{\beta,0}}$).

From the point of view of the classical beta-expansion the second substitution $\sigma'_{\beta,0}$ does not seem to be very interesting. But it becomes important in context with generalised beta-expansions. Let $\delta := 3/(\beta+1)$. Without difficulties we obtain that $\mathfrak{d}_{\beta,\delta}(-\delta) = -2,(-2,-1)^\omega$ and $\mathfrak{d}^*_{\beta,\delta}(1-\delta) = (2,1)^\omega$, therefore the subshift $\Omega_{\beta,\delta}$ is sofic. Condition (5.2) is satisfied and, hence, $\mathfrak{d}^*_{\beta,\delta}(1-\delta) - \mathfrak{d}_{\beta,\delta}(-\delta) = \mathfrak{d}^*_{\beta,0}(1)$ (*cf.* Lemma 5.6). Observe that the minimal periods and pre-periods do not coincide.

Let

$$\sigma_{\beta,\delta} : 1 \mapsto 11211, 2 \mapsto 1311, 3 \mapsto 1121$$

(over the alphabet $\mathcal{A} = \{1,2,3\}$), c_δ the coding prescription determined by

$$c_\delta(1) = \{0,1,2\}, c_\delta(2) = \{0,1\}, c_\delta(3) = \{0,1,2\},$$

and $\mathbf{v} := (1, (\sqrt{21}-3)/2, (\sqrt{21}-3)/2)$ the left eigenvector. Then $\tilde{I}_{\sigma_{\beta,\delta},c}(\bar{1}) = [-\delta, 0)$ and $\tilde{I}_{\sigma_{\beta,\delta},c}(1) = [0, 1-\delta)$ and the respective expansions coincide with the (β,δ)-expansions. We see that $\sigma_{\beta,\delta}$ and $\sigma'_{\beta,0}$ coincide up to the order of the letters but they are not conjugate substitutions as in Example 6.4.

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