Fractal tiles associated to generalised radix representations and shift radix systems

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Shift Radix Systems

Definition

Let $\mathbf{r} \in \mathbb{R}^d$ and

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor).$$

$\tau_{\mathbf{r}}$ is called a shift radix system (SRS) if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

Shift Radix Systems have been introduced by Akiyama *et al.* in 2005

The sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$

$$\begin{aligned}\mathcal{D}_d &:= \{\mathbf{r} \in \mathbb{R}^d \mid \forall \mathbf{x} \in \mathbb{Z}^d \exists n, l \in \mathbb{N} : \tau_{\mathbf{r}}^k(\mathbf{x}) = \tau_{\mathbf{r}}^{k+l}(\mathbf{x}) \ \forall k \geq n\} \\ \mathcal{D}_d^0 &:= \{\mathbf{r} \in \mathbb{R}^d \mid \tau_{\mathbf{r}} \text{ is SRS}\}\end{aligned}$$

For $\mathbf{r} = (r_0, \ldots, r_{d-1})$ denote by $R(\mathbf{r}) \in \mathbb{R}^{d \times d}$ the companion matrix with characteristic polynomial $x^d + r_{d-1}x^{d-1} + \cdots + r_0$. We have

- $\mathcal{D}_d^0 \subset \mathcal{D}_d$,
- $\mathbf{r} \in \mathrm{int}\mathcal{D}_d \Leftrightarrow R(\mathbf{r})$ has spectralradius < 1,
- $\mathcal{D}_d \cap (x = 0) = \mathcal{D}_{d-1}$, $\mathcal{D}_d^0 \cap (x = 0) = \mathcal{D}_{d-1}^0$.

Periodic points

Definition

For $\mathbf{r} \in \mathbb{R}^d$ a point $\mathbf{z} \in \mathbb{Z}^d$ is called purely periodic (with respect to $\tau_{\mathbf{r}}$) if

$$\exists l \in \mathbb{N} : \tau_{\mathbf{r}}^l(\mathbf{z}) = \mathbf{z}.$$

Lemma

For each $\mathbf{r} \in \operatorname{int}\mathcal{D}^d$ there exists only finitely many purely periodic points.

- $\mathbf{r} \in \mathcal{D}_d \Leftrightarrow \forall \mathbf{z} \in \mathbb{Z}^d \exists n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{z})$ is purely periodic.
- $\mathbf{r} \in \mathcal{D}_d^0 \Leftrightarrow \mathbf{0}$ is the only purely periodic point.

$\mathcal{D}_d$ for small d

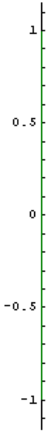


Figure: $\mathcal{D}_1$

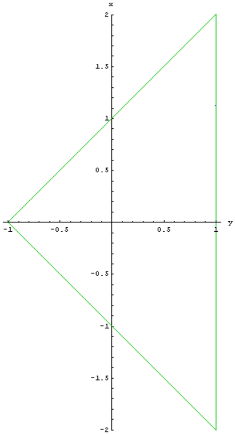


Figure: $\mathcal{D}_2$

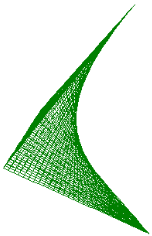


Figure: $\mathcal{D}_3$

$\mathcal{D}_d^0$ for small d

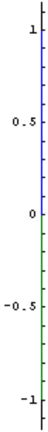


Figure: $\mathcal{D}_1^0$

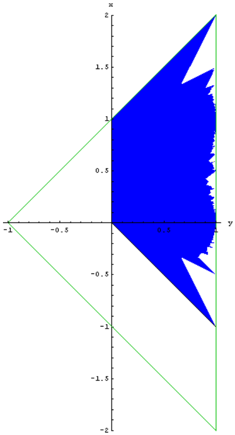


Figure: $\mathcal{D}_2^0$ (approx.)

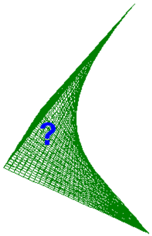


Figure: $\mathcal{D}_3^0$

SRS-tiles

Definition

For $\mathbf{r} \in \mathrm{int}\mathcal{D}_d$, $\mathbf{x} \in \mathbb{Z}^d$ define

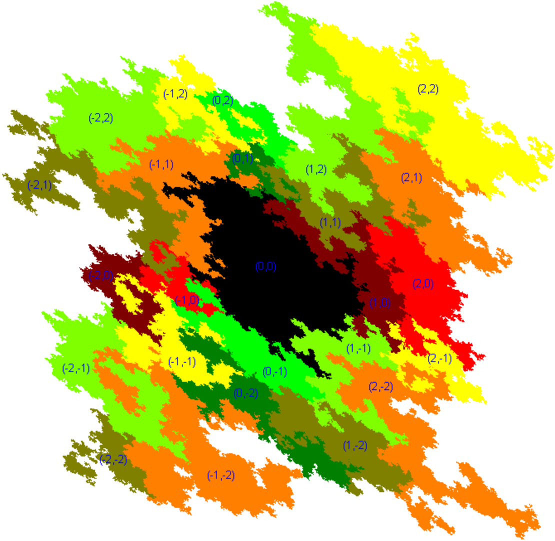
$$T_{\mathbf{r},n}(\mathbf{x}) := \left\{ \mathbf{z} \in \mathbb{Z}^d \,|\tau_{\mathbf{r}}^n(\mathbf{z}) = \mathbf{x} \right\}.$$

The set

$$T_{\mathbf{r}}(\mathbf{x}) = \lim_{n\to\infty} R(\mathbf{r})^n T_{\mathbf{r},n}(\mathbf{x})$$

(limit with respect to some Hausdorff measure) is called the SRS-tile associated to $\mathbf{x}$. $T_{\mathbf{r}}(\mathbf{0})$ is called the central SRS-tile.

SRS-tiles for $\mathbf{r} = (\frac{3}{4}, 1)$



Basic properties of SRS-tiles

- Each SRS-tile is a compact set.
- $\bigcup_{x\in\mathbb{Z}^d} \mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \mathbb{R}^d$.
- $\mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \bigcup_{\mathbf{z}\in T_{\mathbf{r},1}(\mathbf{x})} R(\mathbf{r})\mathcal{T}_{\mathbf{r}}(\mathbf{z})$.
- $\mathbf{0} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x}) \Leftrightarrow \mathbf{x}$ is purely periodic.
- $\tau_{\mathbf{r}}$ is an SRS $\Rightarrow \mathbf{0}$ is an inner point of the central tile.

Tiles associated to an expanding polynomial

Definition (*cf.* Kátai, Kőrnyei)

Let $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ an expanding polynomial ($\Rightarrow |a_0| \geq 2$) and B the transposed companion matrix with characteristic polynomial $A(x)$.

$$\mathcal{F} := \left\{ \mathbf{t} \in \mathbb{R}^d \middle| \mathbf{t} = \sum_{i=0}^{\infty} B^{-i}(c_i, 0, \ldots, 0)^T, c_i \in \mathcal{N} \right\}$$

($\mathcal{N} = \{0, \ldots, |a_0| - 1\}$) is called tile associated to $A(x)$.

Lemma

- *$\mathcal{F}$ is compact and self-affine.*
- *$\mathcal{F}$ is the closure of its interior.*
- $\{\mathbf{x} + \mathcal{F}, \mathbf{x} \in \mathbb{Z}^d\}$ *defines a tiling of* $\mathbb{R}^d$.

Examples of tiles

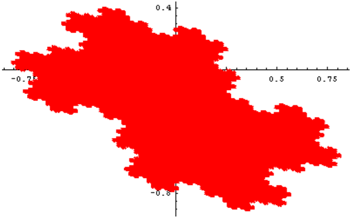


Figure: $\mathcal{F}$ associated to $x^2 - x + 2$

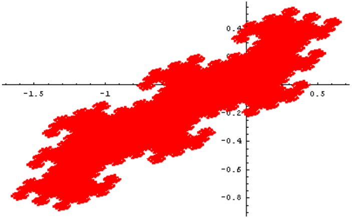


Figure: $\mathcal{F}$ associated to $x^2 + 2x + 3$

Relation to SRS-tiles

$$\mathbf{r} = \left(\frac{1}{a_0}, \frac{a_{d-1}}{a_0}, \dots, \frac{a_1}{a_0}\right), \quad V = \begin{pmatrix} 1 & a_{d-1} & \cdots & a_1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{d-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Theorem

For all $\mathbf{x} \in \mathbb{Z}^d$ *we have*

$$\begin{aligned} \mathcal{F} &= V T_{\mathbf{r}}(\mathbf{0}), \\ \mathbf{x} + F &= V T_{\mathbf{r}}(V^{-1}(\mathbf{x})). \end{aligned}$$

β-expansion

Let $\beta > 1$ a Pisot number with minimal Polynomial $P(x) = x^{d+1} + p_d x^d + \cdots + p_0$. Each $\gamma \in \mathbb{Z}[\frac{1}{\beta}] \cap [0,1)$ has a unique (finite or periodic) representation

$$\gamma = \sum_{i \leq -1} b_i \beta^i, \qquad (b_i \in \{0, \ldots, \lfloor \beta \rfloor\})$$

which satisfies the greedy condition

$$\sum_{i \leq k} b_i \beta^i < \beta^{k+1}, \qquad (\forall k \leq -1).$$

It can be obtained by the β-transformation

$$T_\beta : \gamma \mapsto \gamma\beta - \lfloor \gamma\beta \rfloor.$$

Tiles associated to Pisot numbers

Let
$\beta = \beta_0, \beta_1, \ldots, \beta_d$ the galois conjugates of β, $d = p + 2q$,
$\beta_0, \ldots, \beta_p \in \mathbb{R}$,
$\beta_{p+1} = \overline{\beta_{p+1+q}}, \ldots, \beta_{p+q} = \overline{\beta_{p+2q}} \in \mathbb{C}$,
$\gamma^{(i)}$ the corresponding conjugate of $\gamma \in \mathbb{Q}(\beta)$, $i \in \{0, \ldots, d\}$.

$$\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^d, \gamma \mapsto \begin{pmatrix} \gamma^{(1)} \\ \vdots \\ \gamma^{(p)} \\ \Re(\gamma^{(p+1)}) \\ \Im(\gamma^{(p+1)}) \\ \vdots \\ \Re(\gamma^{(p+q)}) \\ \Im(\gamma^{(p+q)}) \end{pmatrix}$$

Tiles associated to Pisot numbers

Definition (*cf.* Akiyama)

For $\omega \in \mathbb{Z}[\beta] \cap [0,1)$ let

$$\tilde{S}_{\beta,n}(\omega) = \left\{ \gamma \in \mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0,1)] \,\middle|\, T_\beta^n(\gamma) = \omega \right\}.$$

$$\tilde{S}_\beta(\omega) = \lim_{n\to\infty} \Phi(\beta^n \tilde{S}_{\beta,n}(\omega))$$

(with the Hausdorff limit) is called a β-tile.

Lemma

For units we have exactly
$\left|\left\{\gamma \in \mathbb{Z}[\beta] \,\middle|\, \exists n \geq 0 : T_\beta^n(\gamma) = 1-\beta\right\}\right|$ *tiles up to translation.*
Each tile is the closure of its interior.

Examples of β-tiles

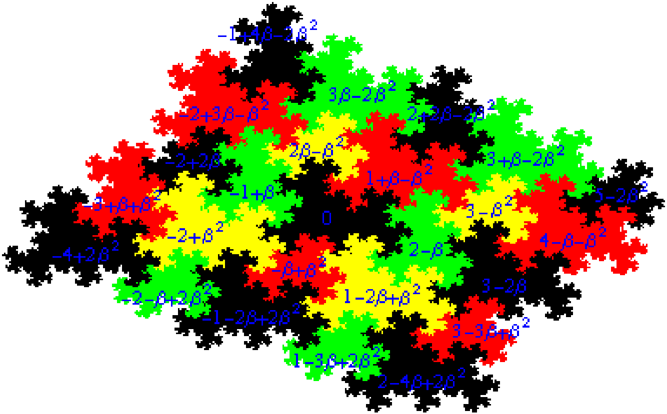


Figure: β-tiles for $\beta = 1.46557\ldots$ $(x^3 - x^2 - 1)$

β-tiles (alternative definition)

Definition

For $\omega \in \mathbb{Z}[\beta] \cap [0,1)$ let

$$S_{\beta,n}(\omega) = \left\{ \gamma \in \mathbb{Z}[\beta] \cap [0,1)] \,\middle|\, T_\beta^n(\gamma) = \omega \right\}.$$

$$S_\beta(\omega) = \lim_{n\to\infty} \Phi(\beta^n S_{\beta,n}(\omega))$$

(with the Hausdorff limit) is called a new β-tile.

Note

$$S_{\beta,n}(\omega) \subseteq \tilde{S}_{\beta,n}(\omega)$$

where equality holds exactly for units.

Examples of new β- tiles

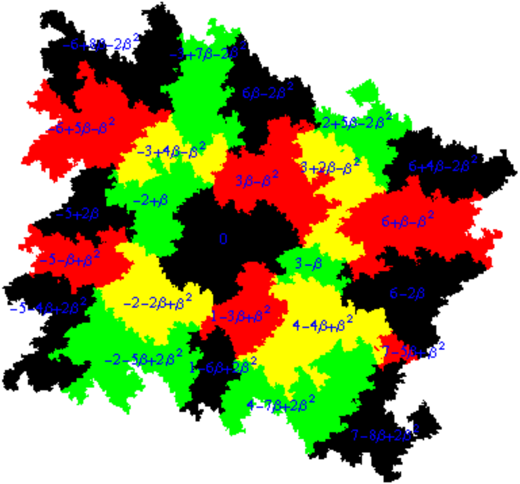


Figure: New β-tiles for $\beta = 2.89329\ldots$ $(x^3 - 3x^2 + x - 2)$

Relation between SRS-tiles and new β-tiles

$r_{d-1} = p_d + \beta, r_{d-2} = p_{d-1} + \beta r_{d-1}, \cdots, r_0 = p_1 + \beta r_1$
$\mathbf{r} = (r_0, \ldots, r_{d-1})$
$\Rightarrow R(\mathbf{r})$ has the (distinct) eigenvalues $\beta_1, \ldots, \beta_d$
$f : \mathbb{Z}^d \to \mathbb{Z}[\beta] \cap [0,1), \mathbf{z} \mapsto \mathbf{rz} - \lfloor \mathbf{rz} \rfloor$ (bijective!)

Theorem

Let $\mathbf{x} \in \mathbb{Z}^d$ and $\omega = f(\mathbf{x})$. There are matrices $U, \Lambda_\beta \in \mathbb{R}^{d\times d}$, $R(\mathbf{r}) = U^{-1}\Lambda_\beta U$, with

$$S_\beta(\omega) = U\left(R(\mathbf{r}) - \beta I_d\right) T_{\mathbf{r}}(\mathbf{x}),$$

where I_d is the d-dimensional identity, and

$$S_\beta(\omega) = \bigcup_{\gamma \in S_{\beta,1}(\omega)} \Lambda_\beta S_\beta(\gamma).$$

Repetition

- For $\mathbf{r}$ induced by an expanding Polynomial each SRS-tile is a translate of the central tile.
- For $\mathbf{r}$ induced by a Pisot unit, there are $\left|\left\{\mathbf{x} \in \mathbb{Z}^d \,\middle|\, \exists n \geq 0 : \tau_{\mathbf{r}}(\mathbf{x}) = (0, \ldots, 0, 1)^T \right\}\right|$ tiles up to translation.
- For other $\mathbf{r}$: ?

Find the differences

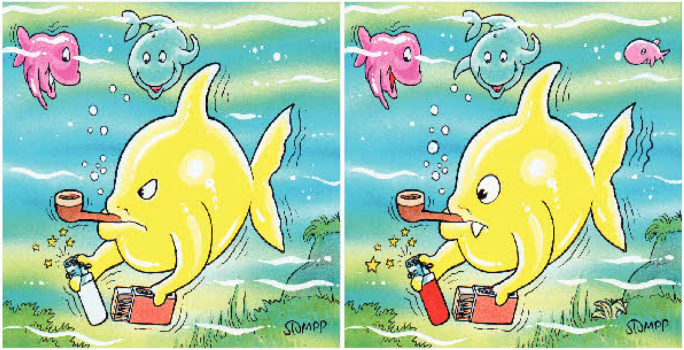


Figure: Find the 10 differences

Find the differences II

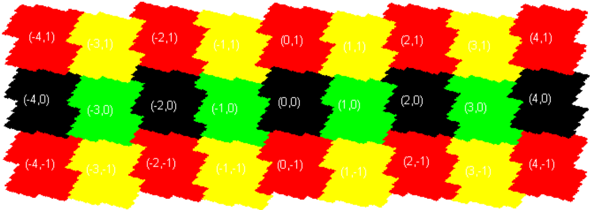
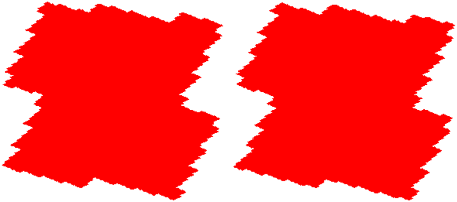


Figure: SRS-tiles for $\mathbf{r} = \left(\frac{1}{4}, \frac{1}{8}\right)$



$T_{\mathbf{r}}(4, -1)$ $T_{\mathbf{r}}(4, 1)$

Tiling

- For $\mathbf{r}$ induced by an expanding Polynomial, SRS-tiles provide a tiling of $\mathbb{R}^d$.
- For other $\mathbf{r}$: There exist no counterexamples that SRS-tiles do not induce a tiling of the $\mathbb{R}^d$.

Closure of the interior

For $\mathbf{r}$ induced by an expanding Polynomial or by a Pisot unit, each SRS-tile is the closure of its interior.

This is NOT true for each $\mathbf{r} \in \mathrm{int}(\mathcal{D}_d)$.

Example

Set $\mathbf{r} = (\frac{9}{10}, -\frac{11}{20})$. The points $\mathbf{z}_0 = (-1,-1), \mathbf{z}_1 = (-1,1), \mathbf{z}_2 = (1,2), \mathbf{z}_3 = (2,1), \mathbf{z}_4 = (1,-1)$ are purely periodic:

$$\tau_{\mathbf{r}} : \mathbf{z}_0 \mapsto \mathbf{z}_1 \mapsto \mathbf{z}_2 \mapsto \mathbf{z}_3 \mapsto \mathbf{z}_4 \mapsto \mathbf{z}_0.$$

But $T_{\mathbf{r},n}(\mathbf{z}_0) = \{\mathbf{z}_{n \mod 5}\}$ (the points form an “isolated period”) and

$$T_{\mathbf{r}}(\mathbf{z}_0) = T_{\mathbf{r}}(\mathbf{z}_1) = T_{\mathbf{r}}(\mathbf{z}_2) = T_{\mathbf{r}}(\mathbf{z}_3) = T_{\mathbf{r}}(\mathbf{z}_4) = \{\mathbf{0}\}.$$

Modern Art

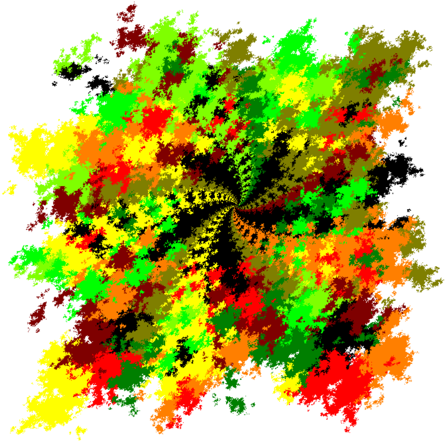
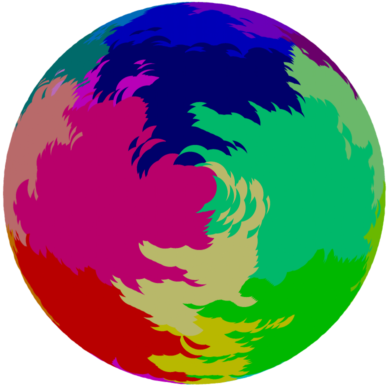


Figure: SRS-tiles for $\mathbf{r} = (\frac{9}{10}, -\frac{11}{20})$

Thanks



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The slides are (soon) available : www.palovsky.com

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