NEW CHARACTERISATION RESULTS FOR SHIFT RADIX SYSTEMS

For $\mathbf{r} \in \mathbb{R}^{\mathbf{d}}$ define the mapping

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \mapsto (x_2, \ldots, x_d, -\lfloor \mathbf{r} \cdot \mathbf{x} \rfloor).$$

$\tau_{\mathbf{r}}$ is called a shift radix system (SRS) if $\forall \mathbf{x} \in \mathbb{Z}^d \, \exists n \in \mathbb{N} : \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}$. Shift radix systems are strongly related to other well known notions of number systems as β-expansion [9, 11] or canonical number systems [10]. Let

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^{\mathbf{d}} \,|\tau_{\mathbf{r}} \text{ is ultimately periodic} \right\} \text{ and}$$
$$\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^{\mathbf{d}} \,|\tau_{\mathbf{r}} \text{ is an SRS} \right\}.$$

Obviously $\mathcal{D}_d^0 \subset \mathcal{D}_d$. The set $\mathcal{D}_d$ is bounded and connected. Its interior can be described relatively easy: for an $\mathbf{r} = \{r_1, \ldots, r_d\} \in \mathbb{R}^d$ define

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & -r_{d-1} & -r_d \end{pmatrix}$$

and the set

$$\mathcal{E}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \,|\rho(R(\mathbf{r})) < 1 \right\},$$

where $\rho(A)$ denotes the spectral radius of the matrix A. Then $\operatorname{int} \mathcal{D}_d = \mathcal{E}_d$ (see [1, section 4]). An analysis of the boundary seems to be difficult and has been done only partially (for $d = 2$, see [1] or [3]). The set $\mathcal{D}_d^0$ can be obtained by cutting out polyhedra (cutout-polyhedra) from $\mathcal{D}_d$. Each of these polyhedra corresponds to a period of $\tau_{\mathbf{r}}$ of integer vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $\tau_{\mathbf{r}} : \mathbf{v}_1 \mapsto \mathbf{v}_1 \mapsto \cdots \mapsto \mathbf{v}_n \mapsto \mathbf{v}_1$. Such a period induces a system of linear inequalities which is sufficient exactly for the corresponding polyhedron. Each closed set $Q \subset \operatorname{int} \mathcal{D}_d$ intersects with only finitely many cutout-polyhedra, but infinitely many cutout-polyhedra are needed to describe $\mathcal{D}_d^0$. The difficulties are at the boundary. Up to now, the 2-dimensional case is the best known one. In [1] and [2] big areas of $\mathcal{D}_2$ have been analysed in order to characterse $\mathcal{D}_2^0$. Especially near the boundary of $\mathcal{D}_2$ we have a very complicated structure. Ideas for algorithms, that can help characterising $\mathcal{D}_d^0$, and some basic applications of them were also presented in [1] and [2]. In [12] these algorithms have been improved and implemented in `Mathematica`®. We

FIGURE 1. An overview of $\mathcal{D}_d^0$

present these results that yield a very good image of the set $\mathcal{D}_2^0$ as it is shown in figure 1. The whole triangle represents the set $\mathcal{D}_2$, the black polygons are cut out. Less than 1.86% (grey) of the entire area of $\mathcal{D}_2$ is left to analyse whether it is part of $\mathcal{D}_2^0$. For the visualization the program *cdd* of Fukuda [6] has been used which converts a given system of inequalities into the list of vertices of the polygon.

Beside algorithmic ways to solve the problem of characterising $\mathcal{D}_2^0$ there are other approaches. From [1, 12] we know two infinite families of cutout-polyhedra. One cuts out triangles, the other one quadrangles from $\mathcal{D}_2$. Each neighbourhood of the point $(1,1)$ intersects with infinitely many polyhedra of the first family, each neighbourhood of $(1,0)$ intersects with infinitely many polyhedra of the second one.

The mentioned algorithms' aim is, to find all the periods that have corresponding cutout-polyhedra within a closed set $Q \subset \mathcal{D}_d$. One of them is based on Brunotte [5]: construct the set $\mathcal{V}(Q) \subset \mathbb{Z}^d$ recursively by observing

$$\begin{aligned}
\mathcal{V}_0(Q) &:= \{\pm(\delta_{1i}, \delta_{2i} \dots, \delta_{di}) | i = 1, \dots, d\}\,, \\
\mathcal{V}_{i+1}(Q) &:= \bigcup_{\mathbf{x} \in \mathcal{V}_i(Q)} \left\{ (x_2, \dots, x_d, j) \,\middle|\, j = \min_{\mathbf{r} \in Q_{\mathbf{x}}} \lfloor -\mathbf{r}\mathbf{x} \rfloor, \dots, \max_{\mathbf{r} \in Q_{\mathbf{x}}} -\lfloor \mathbf{r}\mathbf{x} \rfloor \right\} \\
&\quad \cup \mathcal{V}_i(Q).
\end{aligned}$$

δ_{ji} denotes the Kronecker delta, $\mathbf{x} = (x_1, \dots, x_d)$ and the set $Q_{\mathbf{x}} \subset \partial Q$ consists of the points where $\mathbf{r}\mathbf{x}$ is extreme. For sufficiently small Q this recursion stabilises, i.e. $\exists k : \mathcal{V}_{k+1}(Q) = \mathcal{V}_k(Q)$. Then we set $\mathcal{V}(Q) := \mathcal{V}_k(Q)$. With this set we build up a directed graph $G = V \times E$ with set of vertices $V = \mathcal{V}(Q)$ and edges $E \subset \mathcal{V}(Q) \times \mathcal{V}(Q)$ with

$$(\mathbf{x}, \mathbf{y}) \in E \Leftrightarrow \exists \mathbf{r} \in Q : \tau_{\mathbf{r}}(\mathbf{x}) = \mathbf{y}.$$

Now each period, that induces a cutout-polyhedron intersecting with Q, coresponds to a cycle of this graph. Hence all these periods can be obtained by analyzing the cycles of G. For big Q the set $\mathcal{V}(Q)$ can be infinite. Then Q is subdivided into sufficiently small subsets and the procedure is applied on each of them separately. However, the graph can be very big, especially for Q near the boundary of $\mathcal{D}_d$. Handling them without a computer is nearly impossible.

The mapping $\tau_{\mathbf{r}}$ can be modified in the following way:

$$\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \mapsto (x_2, \ldots, x_d, -\lfloor \mathbf{r} \cdot \mathbf{x} + \frac{1}{2} \rfloor)$$

for an $\mathbf{r} \in \mathbb{R}^{\mathbf{d}}$. If $\forall \mathbf{x} \in \mathbb{Z}^d \, \exists n \in \mathbb{N} : \tilde{\tau}_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}$, we call $\tilde{\tau}_{\mathbf{r}}$ a symmetric shift radix system (SSRS). The sets $\tilde{\mathcal{D}}_d$ and $\tilde{\mathcal{D}}_d^0$ are defined in an analogous manner. Again we have $\mathcal{E}_d \subset \tilde{\mathcal{D}}_d \subset \overline{\mathcal{E}_d}$, but note that $\partial\tilde{\mathcal{D}}_d \neq \partial\mathcal{D}_d$, and analagously $\tilde{\mathcal{D}}_d^0$ can be obtained by cutting out polyhedra from $\tilde{\mathcal{D}}_d$. This symmetric case is interesting because finitely many polyhedra seem to suffice, at least for small d . Akiyama and Scheicher [4] analysed the case $d = 2$ and completely characterised the set $\tilde{\mathcal{D}}_2^0$. It is a triangle with two lines of the boundary removed. The three dimensional case is a little more complex. The analysis of $\tilde{\mathcal{D}}_3^0$ requires the support of the computer by using an adapted version of the above algorithm. As result we gain a rather simple figure, a composition of three convex bodies.

Supported by FWF Project Nr. P9610-N13.

References

[1] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. M. Thuswaldner, *Generalized radix representations and dynamical systems I*, Acta Math. Hungar. 108 (2005), no. 3, 207–238.

[2] S. Akiyama, H. Brunotte, A. Pethő and J. M. Thuswaldner, *Generalized radix representations and dynamical systems II*, Acta Arithmetica 121 (2006), no. 1, 21–61.

[3] S. Akiyama, H. Brunotte, A. Pethő and W. Steiner, *Remarks on a conjecture on certain integer sequences*, Period. Math Hungar., to appear

[4] S. Akiyama, K. Scheicher, *Symmetric shift radix systems and finite expansions*, preprint

[5] H. Brunotte, *On trinomial bases of radix representations of algebraic integers*, Acta Sci. Math. Acta Sci. Math. (Szeged) 67 (2001), no. 3-4, 521–527.

[6] K. Fukuda, *cdd and cddplus Homepage*, ETHZ, Zürich, Switzerland, `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/index.html`.

[7] T. S. Motzkin, H. Raiffa, G. L. Thompson,R. L. Thrall, *The double description method* Contributions to the theory of games, vol. 2, pp. 51-73. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., (1953).

[8] S. Pemmaraju, S. Skiena, *Computational Discrete Mathematics, Combinatorics and Graph Theory with Mathematica®*, Cambridge University Press (2003).
[9] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.
[10] A. Pethő, *On a polynomial transformation and its application to the construction of a public key cryptosystem*, in: Computational Number Theory, A. Pethő, M. Pohst, H. C. Williams and H. G. Zimmer (eds.), de Gruyter (1991), 31–43.
[11] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. 8 (1957), 477–493.
[12] P. Surer, *New characterisation results for shift radix systems*, preprint
[13] R. Tarjan, *Depth-first search and linear graph algorithms*, SIAM J. Comput. 1 (1972), no. 2, 146-160.