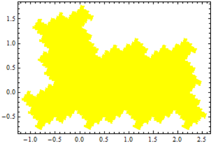
Coding of substitution dynamical systems

Paul Surer

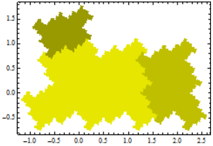
Departamento de Matemática
Instituto de Biociências, Letras e Ciências Exatas (IBILCE)
Universidade Estadual Paulista (UNESP)
São José do Rio Preto - SP - Brazil

Liège, June 2011

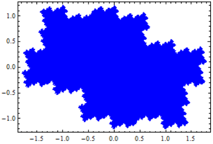
Central tile induced by the beta-expansion with β the real root of $x^3 - 2x^2 - 1$.



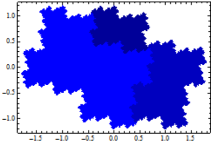
Rauzy fractal associated with the substitution $1 \mapsto 112, 2 \mapsto 3, 3 \mapsto 1$.



Central tile induced by the symmetric beta-expansion with β the real root of $x^3 - 2x^2 - 1$.



Rauzy fractal associated with the substitution $1 \mapsto 121, 2 \mapsto 3, 3 \mapsto 1$.



Possible Explanations

- ~~Paul messed up with Mathematica®~~.
- ~~There is a accidental similarity~~.
- There is an ~~unknown~~ relation between the shift of finite type induced by the symmetric beta expansion and the substitution dynamical system.

A basic result

Theorem (*cf.* Mossé, Durand et al., Canterini-Siegel, Holton-Zamboni, ...)

For every $w \in \Omega$ *there is a unique pair* $(u, k) \in \Omega \times \mathbb{N}$ *with* $0 \leq k < |\sigma(u_0)|$ *such that* $w = S^k \sigma(u)$.

Corollary

For every $w \in \Omega$ *there is a unique pair* $(u', k')\Omega \times \mathbb{Z}$ *with* $-|\sigma(u_{-1})| < k' \leq 0$ *such that* $w = S^{k'} \sigma(u)$.

Idea

We combine these two assertions. To maintain uniqueness we use a coding prescription.

Coding prescription

Definition (Coding prescription)

A coding prescription is a collection $(\mathcal{C}(j))_{j\in\mathcal{A}}$ where each $\mathcal{C}(j)$ is a complete set of residues modulo $|\sigma(j)|$ whose entries are smaller than $|\sigma(j)|$ in modulus.

Notation

For a coding prescription $\mathcal{C}$ we denote for each $j \in \mathcal{A}$ the non-negative elements of $\mathcal{C}(j)$ by $\mathcal{C}(j)^+$ and the non-positive elements by $\mathcal{C}(j)^-$.

Theorem

Let $\mathcal{C}$ a coding prescription. Then for every $w \in \Omega$ there is a unique pair $(\Theta(w), \phi(w)) \in \Omega \times \mathbb{Z}$ such that $\phi(w) \in \mathcal{C}(\Theta(w)_{-1})^- \cup \mathcal{C}(\Theta(w)_0)^+$ and $w = S^{\phi(w)}\sigma(\Theta(w))$.

Inverse letters

Notation

Let $\tilde{\mathcal{A}} := \mathcal{A} \cup \{\bar{j} | j \in \mathcal{A}\}$ and $\tilde{\mathcal{A}}^*$ the finite words over $\tilde{\mathcal{A}}$.

Rules

Observe the following rules and notations: for all $A, B \in \tilde{\mathcal{A}}^*$ we have

- $\overline{AB} = \overline{B}\,\overline{A}$;
- $\overline{(\overline{A})} = A$ for all $A \in \tilde{\mathcal{A}}^*$;
- $A\overline{A} = \varepsilon$ for all $A \in \tilde{\mathcal{A}}^*$;
- $\overline{\varepsilon} = \varepsilon$;
- $\sigma(\overline{A}) = \overline{\sigma(A)}$.

"Digits expansion" for substitutions

Digit set

$\mathcal{R} = \mathcal{R}(\mathcal{C}) := \{(a.b, d) |\ ab \in \mathcal{L}_2, \exists k \in \mathcal{C}(a)^- \cup \mathcal{C}(b)^+, \sigma(ab)_{[1,|\sigma(a)|+k]} = \sigma(a)d\}$.

Expansion

$\Gamma : \Omega \longrightarrow \mathcal{R}^\infty, w \mapsto (\Theta^n(w)_{-1}.\Theta^n(w)_0, d_n)_{n\geq 1}$

where, for every $n \geq 1$, d_n is the (uniquely determined) element of $\tilde{\mathcal{A}}^*$ with $\Theta^{n-1}(w) = \sigma(\Theta^n(w)_{(-\infty,-1]})d_n.\overline{d_n}\sigma(\Theta^n(w)_{[0,\infty)})$.

Note

- d_n consists of negative or positive letters only for all $n \in \mathbb{N}$.
- With the convention that inverse letters give negative contributions to the length of a word we have $|d_n| = \phi(\Theta^{n-1}(w))$.

Properties

Proposition

Let $\Gamma(w) = (a_n.b_n, d_n)_{n\geq 1}$. If for infinitely many indices n we have $|\sigma(a_n)d_n| > 1$ and $|\overline{d_n}\sigma(b_n)| > 1$, respectively then $w = \lim_{n\to\infty} \sigma^n(a_n)\sigma^{n-1}(d_n)\cdots d_1.\overline{d_1}\cdots\sigma^{n-1}(\overline{d_n})\sigma^n(b_n)$.

Proposition

w is a periodic point if and only if $\Gamma(w) = (a_n.b_n, d_n)_{n\geq 1}$ with $d_n = \varepsilon$ for all $n \geq 1$.

Graph representation

Definition

The *coding graph with respect to* $\mathcal{C}$, denoted by $G(\sigma, \mathcal{C})$, is the graph with the following properties: The vertices are pairs $(a.b)$ such that $ab \in \mathcal{L}_2$. There is an edge from $(a.b)$ to $(a'.b')$ labelled by $(a'.b', d) \in \mathcal{R}$ if $(\sigma(a')d)_{|\sigma(a')d|} = a$ and $(\overline{d}\sigma(b'))_1 = b$.

Theorem

Γ maps Ω surjectively onto the infinite paths of $G(\sigma, \mathcal{C})$. Γ is one-to-one with the shift orbit of periodic points as possible exceptions (finite-to-one).

Partial ordering on the edges

Definition

A coding prescription $\mathcal{C}$ is called *continuous* if $\mathcal{C}(j)$ consists of consecutive integers for all $j \in \mathcal{A}$.

Partial ordering

The partial ordering on $\mathcal{R}$
$(a.b, d) \prec (a'.b', d')$ if and only if $a = a', b = b', |d| < |d'|$
induces a partial ordering on the paths.

Note

For a continuous coding prescription the successor map is a "nice" adic transformation. For non-continuous coding prescription the successor map is difficult to describe.

The shift map

Theorem

The successor map on the paths is conjugated to the shift map on Ω.

Theorem

If $\Gamma(w)$ is a maximal (minimal, respectively) element (w.r.t. $\prec$) then $\Gamma(S(w))$ is a minimal element ($\Gamma(S^{-1}(w))$ is a minimal element, respectively).

Symmetric beta-expansions

Conjecture

Let $\beta > 1$ and $d(-\frac{1}{2}) = -t_1, \ldots, -t_q, -t_{q+1}, \ldots, -t_{q+p}$ such that $t_n \geq 0$ for all $n \in \{1, \ldots, p+q\}$. Then the symmetric beta-shift is a factor of the substitution dynamical system induced by

$$\begin{aligned} 1 &\mapsto 1^{t_1} 2 1^{t_1}, \\ &\vdots \\ (p+q-1) &\mapsto 1^{t_{p+q-1}} 2 1^{t_{p+q-1}}, \\ (p+q) &\mapsto 1^{t_{p+q-1}} (q+1) 1^{t_{p+q-1}}. \end{aligned}$$

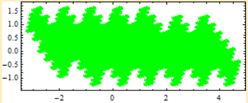
Epsilon-beta-expansions

S. (2009), Kalle-Steiner (2011)

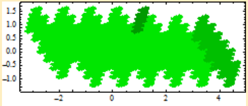
Beta-expansions of the interval $[-\varepsilon, 1-\varepsilon)$ with respect to the transformation $x \mapsto x\beta - \lfloor x\beta + \varepsilon \rfloor$ $(\varepsilon \in [0,1))$.

Example: $\beta^6 - 6\beta^2 - 3\beta - 1 = 0$, $\varepsilon = \frac{1}{3}$

The induces central tile



The Rauzy fractal induced by $1 \mapsto 1111611, 2 \mapsto 1131, 3 \mapsto 1$.



Epsilon-beta-expansions

Conjecture

Let $\beta > 1$, $d(-\varepsilon) = -t_1, \ldots, -t_q, -t_{q+1}, \ldots, -t_{q+p}$ and $d(1-\varepsilon) = s_1, \ldots, s_q, s_{q+1}, \ldots, s_{q+p}$ such that $t_n, s_n \geq 0$ for all $n \in \{1, \ldots, p+q\}$. Then the symmetric beta-shift is a factor of the substitution dynamical system induced by

$$\begin{aligned} 1 &\mapsto 1^{t_1} 2 1^{s_1}, \\ &\vdots \\ (p+q-1) &\mapsto 1^{t_{p+q-1}} 2 1^{s_{p+q-1}}, \\ (p+q) &\mapsto 1^{t_{p+q-1}} (q+1) 1^{s_{p+q-1}}. \end{aligned}$$

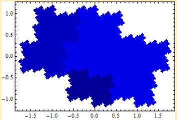
Rauzy fractals

Observation

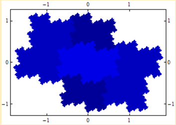
The infinite walks with initial point $a.b$ of the coding graph correspond to the intersection of the bth subtile with the ath subtile under the domain exchange map.

$\sigma : 1 \mapsto 121, 2 \mapsto 3, 3 \mapsto 1$

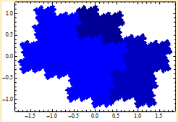
Rauzy fractal under the domain exchange



Intersection



Rauzy fractal



Further ideas

- Consider applications on Automorphisms of the free group (links to other shapes of epsilon-beta-expansions or negative beta-expansions);
- Comparing (intersecting) Rauzy fractals for different substitution that have the same incidence matrix;
- Studying tilings.

Induced substitution

Observation

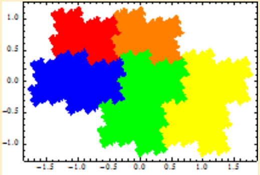
The partial ordering on the edges of the coding graph induces another primitive substitution over $\mathcal{L}_2$. Thus, each continuous coding prescription $\mathcal{C}$ gives an *induced substitution* $\sigma_{\mathcal{C}}$.

- $\sigma_{\mathcal{C}}$ has the same dominant eigenvalue as the original substitution σ;
- $\sigma_{\mathcal{C}}$ is always reducible;
- the right eigenvector of the incidence matrix does not depend on $\mathcal{C}$.

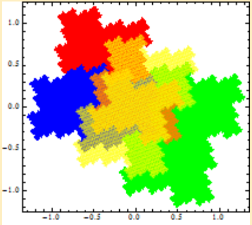
Induced substitution

$\sigma : 1 \mapsto 121, 2 \mapsto 3, 3 \mapsto 1$

$\mathcal{C}_I(1) = \{-2, -1, 0\}$,
$\sigma_{\mathcal{C}_I} : A \mapsto BCA,$
$B \mapsto BCD,$
$C \mapsto E,$
$D \mapsto BCA,$
$E \mapsto A$



$\mathcal{C}_I(1) = \{-1, 0, 1\}$,
$\sigma_{\mathcal{C}_I} : A \mapsto CAB,$
$B \mapsto CD,$
$C \mapsto EB,$
$D \mapsto CA,$
$E \mapsto AB$



$\mathcal{C}_I(1) = \{0, 1, 2\}$,
$\sigma_{\mathcal{C}_I} : A \mapsto ABC,$
$B \mapsto D,$
$C \mapsto EBC,$
$D \mapsto A,$
$E \mapsto ABC$

Thanks

Thank you
for your attention

The research was supported by FAPESP, Proc. 2009/07744-0

E-mail: me@palovsky.com
Homepage : www.palovsky.com