Dynamical properties of the tent map

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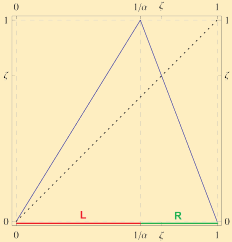
Tent map

Definition

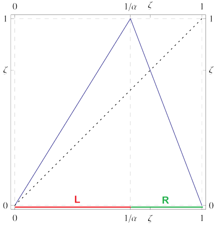
Let $\alpha > 1$ and $\beta = \beta(\alpha) = \frac{\alpha}{\alpha-1}$ (hence, $\alpha^{-1} + \beta^{-1} = 1$). The tent-map T_α is defined by

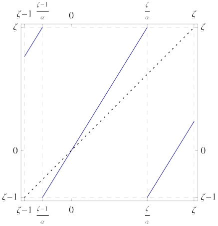
$$T_\alpha : [0,1] \to [0,1], x \longmapsto \begin{cases} \alpha x & \text{if } x \in L := [0, \alpha^{-1}], \\ \beta(1-x) & \text{if } x \in R := (\alpha^{-1}, 1]. \end{cases}$$



Program for the next 20 minutes

- Summarise several well-known results obtaind by Lagarias, Porta and Stolarsky (1993/1994).
- Present new results concerning the structure of orbits ("Finiteness and periodicity properties").
- Sketch a remarkable connection between the dynamics induced by tent maps and by a class of modified beta-transformations.





Basic properies

- Each $x \in [0,1]$ has a unique representation as

$$x = \sum_{0 \leq k < M} \alpha^{-m_k} (-\beta)^{-k}$$

where $M \in \mathbb{N} \cup \{\infty\}$ and $(m_k)_{0 \leq k < M}$ is a (finite or infinite) increasing sequence of non-negative integers.

- Each $x \in [0,1]$ can be uniquely coded as an infinite $\{L, R\}$-sequence $d(x)$.
- The shift space

$$W := \overline{\{d(x) : x \in [0,1]\}}$$

(the closure w.r.t. the product topology) induced by the tent map T_α is the full binary shift $\{L, R\}^{\mathbb{N}}$.

Finiteness property and periodicity property

Definition

Let $\alpha > 1$ (algebraic).

- We say that T_α has the **periodicity property (P)** if for each $x \in [0,1] \cap \mathbb{Q}(\alpha)$ the orbit $(T_\alpha^k(x))_{k \geq 0}$ is eventually periodic.
- The tent map T_α has the **finiteness property (F)** if for each $x \in [0,1] \cap \mathbb{Z}[\alpha^{-1}]$ there exists a k such that $T_\alpha^k(x) = 0$.

Periodicity - Summary of earlier results

Theorem (Lagarias *et al.*)

Let $\alpha > 1$ such that (P) holds. Then α and $\beta(\alpha)$ are both algebraic integers and $\min\{|\alpha'|, |\beta(\alpha')|\} < 1$ holds for all Galois conjugates $\alpha' \neq \alpha$.

Theorem (Lagarias *et al.*)

Let α be a special Pisot number (that is α and $\beta(\alpha)$ are both Pisot numbers). Then (P) holds.

Conjecture (Lagarias *et al.*)

An algebraic integer $\alpha > 1$ satisfies (P) if and only if $|\alpha'||\alpha' - 1|^{1/\alpha - 1} < 1$ holds for all Galois conjugates $\alpha' \neq \alpha$.

Special Pisot numbers

Theorem (Smyth, 1999)

There only exist 11 special Pisot numbers. Additionally, there exists exactly one Pisot number γ such that $\gamma' = \gamma(\gamma-1)^{-1}$ is a Salem number.

Number	Approx. value	Minimal polynomial
α_{-5}	$4.0796\ldots$	$x^3 - 5x^2 + 4x - 1$
α_{-4}	$3.62966\ldots$	$x^4 - 5x^3 + 6x^2 - 4x + 1$
α_{-3}	$3.1479\ldots$	$x^3 - 4x^2 + 3x - 1$
α_{-2}	$2.61803\ldots$	$x^2 - 3x + 1$
α_{-1}	$2.32472\ldots$	$x^3 - 3x^2 + 2x - 1$
α_0	2	$x - 2$
α_1	$1.75488\ldots$	$x^3 - 2x^2 + x - 1$
α_2	$1.61803\ldots$	$x^2 - x - 1$
α_3	$1.46557\ldots$	$x^3 - x^2 - 1$
α_4	$1.38028\ldots$	$x^4 - x^3 - 1$
α_5	$1.32472\ldots$	$x^3 - x - 1$
γ	$1.86676\ldots$	$x^4 - 2x^3 + x - 1$
γ'	$2.15372\ldots$	$x^4 - 3x^3 + 3x^2 - 3x + 1$

Periodicity - A new result

Theorem (S-S-S)

The tent map T_α satisfies the periodicity property (P) for $\alpha = \gamma$ as well as for $\alpha = \gamma'$.

The method for the proof comes from K. Schmidt (1980).

Finiteness - Earlier results and obvious results

Proposition (Lagarias *et al.*)

For $\alpha = \alpha_0 = 2$ or $\alpha = \alpha_2$ (golden mean) the tent map T_α has the finiteness property (F).

Proposition

For $\alpha \in \{\alpha_{-5}, \alpha_{-4}, \alpha_{-2}, \alpha_4, \gamma, \gamma'\}$ (F) does not hold.

Number	Approx. value	Minimal polynomial	(F)
α_{-5}	4.0796 . . .	$x^3 - 5x^2 + 4x - 1$	↯
α_{-4}	3.62966 . . .	$x^4 - 5x^3 + 6x^2 - 4x + 1$	↯
α_{-3}	3.1479 . . .	$x^3 - 4x^2 + 3x - 1$	
α_{-2}	2.61803 . . .	$x^2 - 3x + 1$	↯
α_{-1}	2.32472 . . .	$x^3 - 3x^2 + 2x - 1$	
α_0	2	$x - 2$	✓
α_1	1.75488 . . .	$x^3 - 2x^2 + x - 1$	
α_2	1.61803 . . .	$x^2 - x - 1$	✓
α_3	1.46557 . . .	$x^3 - x^2 - 1$	
α_4	1.38028 . . .	$x^4 - x^3 - 1$	↯
α_5	1.32472 . . .	$x^3 - x - 1$	
γ	1.86676 . . .	$x^4 - 2x^3 + x - 1$	↯
γ'	2.15372 . . .	$x^4 - 3x^3 + 3x^2 - 3x + 1$	↯

Beta-transformations - Preparations

Let $\alpha > 1$, $\zeta = \zeta(\alpha) = \frac{\alpha}{2\alpha-1} = \frac{\beta(\alpha)}{\beta(\alpha)+1}$ (fixed point of the tent map T_α) and $I = I(\alpha) = [\zeta - 1, \zeta)$.

Denote by $\phi : \mathbb{R} \longrightarrow I$ the generalised fractional part: $\phi(x) = x - \lfloor x + (1 - \zeta) \rfloor$ for all $x \in \mathbb{R}$.

Consider the generalised beta-transformation (with respect to the base α and the generalised fractional part ϕ)

$$\tau_\alpha : I \longrightarrow I, \xi \longmapsto \phi(\alpha\xi).$$

Beta-transformations have been studied in a very general way by Kalle, Steiner (2012).

A family of parameters

We are interested in the family

$$\mathfrak{S} := \{\alpha \in (1,2] : \exists P \in \mathbb{N} : \alpha^P = \beta(\alpha)^2\}.$$

Note that $\alpha_i \in \mathfrak{S}$ for $i \in \{0, \ldots 5\}$.
For $\alpha \in \mathfrak{S}$ we can associate to each $x \in I$ a digit string $\delta(x) \in \{-1, 0, 1\}^{\mathbb{N}}$ such that the kth digit equals $\lfloor \alpha \tau_\alpha^k(x) + (1 - \zeta) \rfloor$.

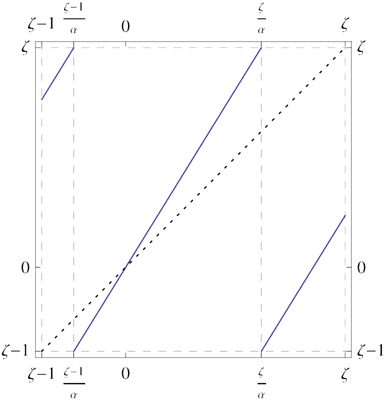
Proposition

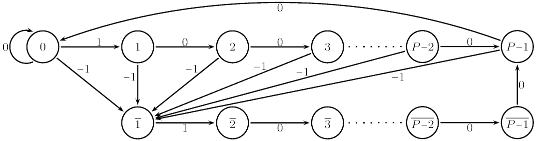
For $\alpha \in \mathfrak{S}$ the subshift

$$\Omega := \overline{\{\delta(x) : x \in [0,1]\}} \subset \{-1, 0, 1\}^{\mathbb{N}}$$

is a shift of finite type.

Transformation and presentation graph





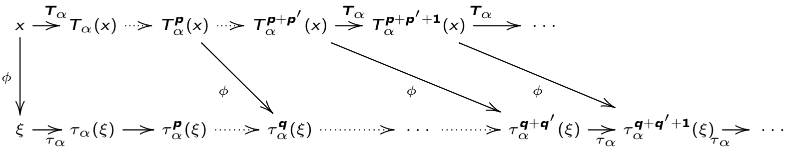
An interesting connection

Theorem (S-S-S)

Let $\alpha \in \mathfrak{S}$. Then for each $x \in [0,1]$ there exists a pair $(p,q) = (p(x), q(x))$ of positive integers such that $\phi \circ T_\alpha^p(x) = \tau_\alpha^q \circ \phi(x)$.

Note

For each $N \in \mathbb{N}$ we can find an (open) interval $J \subset [0,1]$ such that $\phi \circ T_\alpha^p(x) \neq \tau_\alpha^q \circ \phi(x)$ for all $p, q \leq N$.



Consequences I

Theorem (S-S-S)

Let $\alpha \in \mathfrak{S}$ and $x \in [0,1]$.

1. *The T_α-orbit of x is eventually periodic if and only if the τ_α-orbit of $\phi(x)$ is eventually periodic.*
2. *The T_α-orbit of x contains 0 if and only if the τ_α-orbit of $\phi(x)$ contains 0.*

Number	Approx. value	Minimal polynomial	(F)
α_{-5}	$4.0796\ldots$	$x^3 - 5x^2 + 4x - 1$	↯
α_{-4}	$3.62966\ldots$	$x^4 - 5x^3 + 6x^2 - 4x + 1$	↯
α_{-3}	$3.1479\ldots$	$x^3 - 4x^2 + 3x - 1$	
α_{-2}	$2.61803\ldots$	$x^2 - 3x + 1$	↯
α_{-1}	$2.32472\ldots$	$x^3 - 3x^2 + 2x - 1$	
α_0	2	$x - 2$	✓
α_1	$1.75488\ldots$	$x^3 - 2x^2 + x - 1$	✓
α_2	$1.61803\ldots$	$x^2 - x - 1$	✓
α_3	$1.46557\ldots$	$x^3 - x^2 - 1$	✓
α_4	$1.38028\ldots$	$x^4 - x^3 - 1$	↯
α_5	$1.32472\ldots$	$x^3 - x - 1$	✓
γ	$1.86676\ldots$	$x^4 - 2x^3 + x - 1$	↯
γ'	$2.15372\ldots$	$x^4 - 3x^3 + 3x^2 - 3x + 1$	↯

Consequences II

$$\begin{array}{ccc} W = \{L, R\}^{\mathbb{N}} & \xrightarrow{\quad\Psi\quad} & \Omega \\ \uparrow d & & \uparrow \delta \\ [0,1] & \xrightarrow{\quad\phi\quad} & I = [\zeta - 1, \zeta) \end{array}$$

Our setting induces in a natural way a function $\Psi : W \longrightarrow \Omega$ that satisfies

$$\Psi \circ d(x) = \delta \circ \phi(x).$$

for all $x \in [0, 1]$.

Theorem

The function Ψ is not given by an n-block code and cannot be described by a finite state transducer.

Preview - Tiles induced by special Pisot numbers

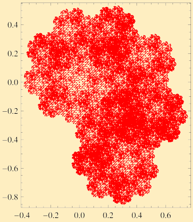
Example

Let $\alpha = \alpha_1 \in \mathfrak{S}$ and $\alpha' \in \mathbb{C}$ one of its Galois conjugates. Then

$$f_L : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto \alpha' z \quad (1)$$

$$f_R : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto \beta(\alpha')(1 - z) \quad (2)$$

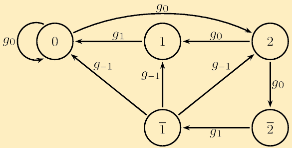
are contractions on the complex plane and the set $\{f_L, f_R\}$ induces an iterated function system (IFS).

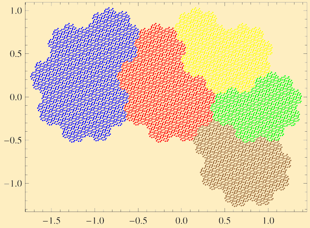


Preview - Tiles induced by beta-transformations

Example

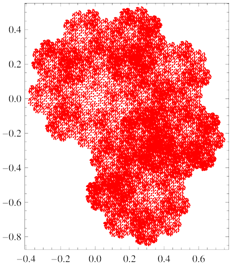
Beta-transformations with respect to Pisot units induce a graph directed iterated function system (GDIFS, see Kalle, Seiner, 2012). In our case (τ_α with $\alpha = \alpha_1$) we obtain.

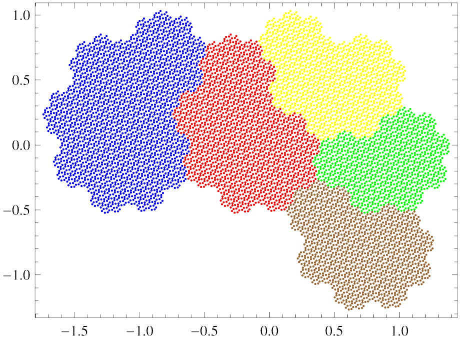




$$g_k(i) = \alpha' z + i$$

Compare the tiles!





Conclusion

Open problems and Qustions

- Does T_α for $\alpha \in \{\alpha_{-1}, \alpha_{-3}\}$ satisfy the finiteness property (F)?
- Does theoretical computer science provide a suitable model for describing Ψ?
- Is there any example for a similar "weak conjugacy" in other contexts?

Thank you for your attention!
Merci pour votre attention!

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