Substitutions, Shifts of finite type, and Numeration

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Der Wissenschaftsfonds.

Preliminaries

Let $\mathcal{A} := \{1, 2, \ldots, m\}$ be a finite set (alphabet) and $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ the free monoid over $\mathcal{A}$ (the empty word ε is the neutral element). For an element of $\mathcal{A}^*$ we denote by $|A|$ the length of A.

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Consider a non-erasing morphism (substitution) $\sigma : \mathcal{A}^* \longmapsto \mathcal{A}^*$.
We call σ primitive if there exists a positive integer n such that b appears in $\sigma^n(a)$ for each pair $(a, b) \in \mathcal{A}^2$.

Graphs associated with substitutions

Substitutions are frequently represented by a finite directed graph. This graph appears under different names and in slightly different forms (labels, edge directions). For example, let $\mathcal{A} = \{1, 2, 3\}$ and $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

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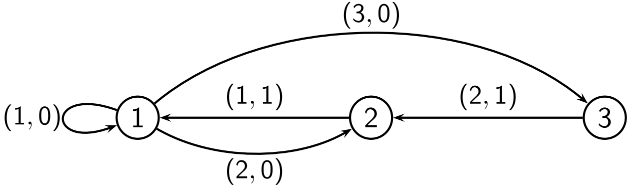

Holton-Zambioni (2001)

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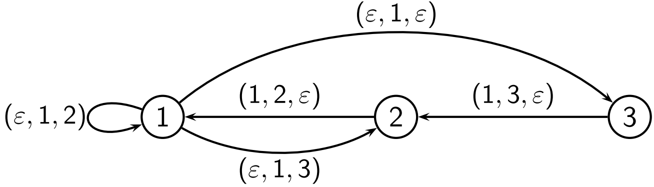

Prefix-Suffix graph, Canterini-Siegel (2001)

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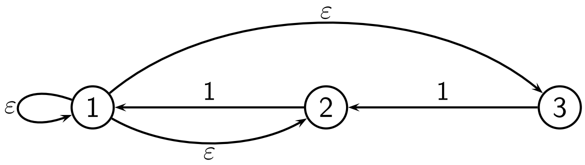

Prefix graph (Rauzy fractals)

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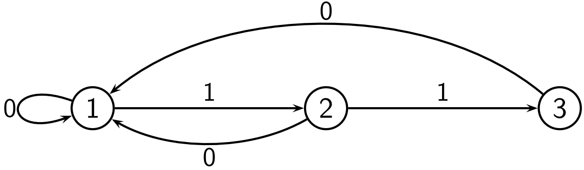

Dumont-Thomas numeration

Periodic points

Let σ be a primitive substitution over $\mathcal{A}$. We may extend σ to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$:

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with $w'_0 w'_1 w'_3 \cdots = \sigma(w_0)\sigma(w_1)\cdots$, and $\cdots w'_{-3} w'_{-2} w'_{-1} = \cdots \sigma(w_{-2})\sigma(w_{-1})$.

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$\cdots w'_{-3} w'_{-2} w'_{-1} = \cdots \sigma(w_{-2})\sigma(w_{-1})$.
We mark this centre by a point and write

$$\sigma(\cdots w_{-2} w_{-1} . w_0 w_1 w_2 w_3 \cdots) = \cdots \sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\ldots.$$

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$$\sigma(\cdots w_{-2} w_{-1}.w_0 w_1 w_2 w_3 \cdots) = \cdots \sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\ldots.$$

There exists at least one *periodic point* $u \in \mathcal{A}^{\mathbb{N}}$, that is $\sigma^n(u) = u$ for some integer $n > 1$. For avoiding trivialities we suppose that u is not periodic. Let

$$\mathfrak{L} := \{A \in \mathcal{A}^* : A \text{ is a subword of } u\}$$

the language induced by σ. Due to primitivity $\mathfrak{L}$ does not depend on the actual choice of u.

A dynamical system

Let

$$\Omega := \{w \in \mathcal{A}^{\mathbb{Z}} : \text{each subword of } w \text{ is contained in } \mathfrak{L}\}$$

and $S : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ the left shift, *i.e.* $S : (w_j)_{j\in\mathbb{Z}} \longmapsto (w_{j+1})_{j\in\mathbb{Z}}$.
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- Ω always contains at least one element fixed by some power of σ. But observe that not each bi-infinite word fixed by some power of σ is contained in Ω.
- A substitution dynamical system is minimal and uniquely ergodic.

Desubstitution

Theorem (*cf.* Mossé (1992))

Let (Ω, S) *be the substittution dynamical system induced by* σ *and* $w \in \Omega$. *Then there exists a uniquely determined* $w' = (w'_j)_{j\in\mathbb{Z}} \in \Omega$ *and a uniquely determined integer* $k \in \{0, \ldots, |\sigma(w'_0)| - 1\}$ *such that*

$$w = S^k \sigma(w').$$

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By successively applying the theorem on $w = w^{(0)} \in \Omega$ we obtain sequences $(w^{(n)})_{n\geq 1} \in \Omega^{\mathbb{N}}$ and $(k_n)_{n\geq 1} \in \mathbb{Z}^{\mathbb{N}}$ with $0 \leq k_n < |\sigma(w_0^{(n)})|$ for all $n \geq 1$ such that $w^{(n-1)} = S^{k_n}\sigma(w^{(n)})$ (where $w_0^{(n)}$ denotes the letter indexed by 0).

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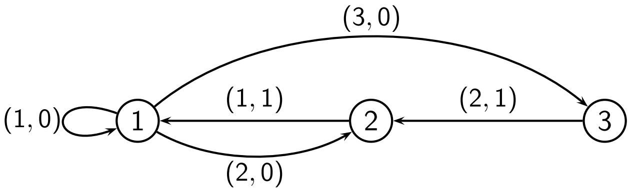
Define

$$\Gamma(w) := (w_0^{(n)}, k_n) \in (\mathcal{A} \times \mathbb{Z})^{\mathbb{N}}.$$

The image $\Gamma(w)$ corresponds to an infinite walk on a finite graph.

Desubstitution

index	$\cdots$	$\overset{-}{7}$	$\overset{-}{6}$	$\overset{-}{5}$	$\overset{-}{4}$	$\overset{-}{3}$	$\overset{-}{2}$	$\overset{-}{1}$	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$

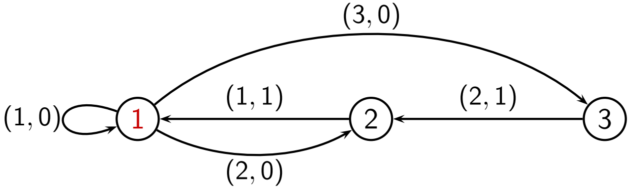


$\Gamma(w_0) =$

Desubstitution

index	⋯	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	⋯
$w^{(0)} =$	⋯	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	⋯

$k = 0$



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Desubstitution

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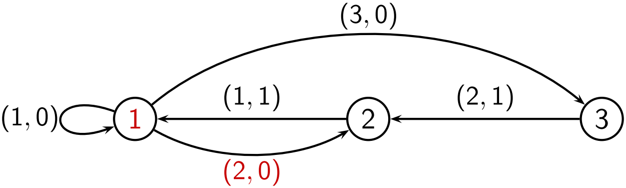
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$									2								



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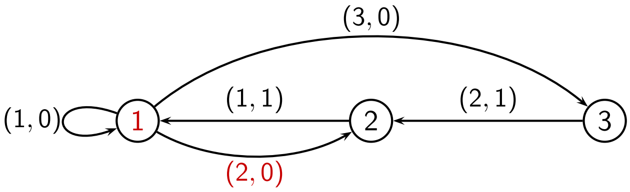
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$								1	2	1							



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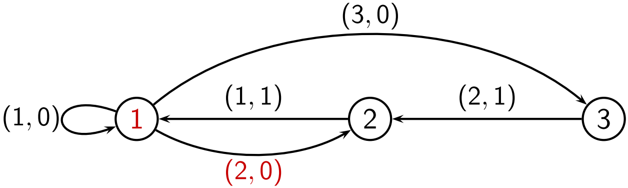
$w^{(0)} =$	···	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	···
									$k = 0$								
$w^{(1)} =$							3	1	2	1	1						



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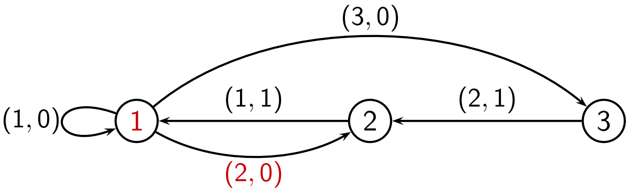
$w^{(0)} =$	⋯	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	⋯
									$k = 0$								
$w^{(1)} =$						1	3	1	2	1	1	2					



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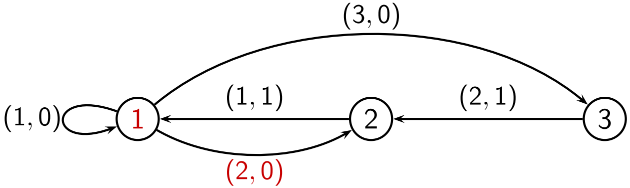
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	2	1	1	2	1	3	1	2	$\cdots$



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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
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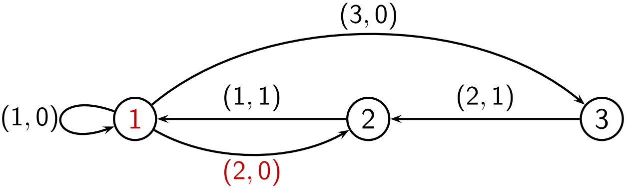

$\Gamma(w_0) = (2, 0),$

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	2	1	1	2	1	3	1	2	$\cdots$
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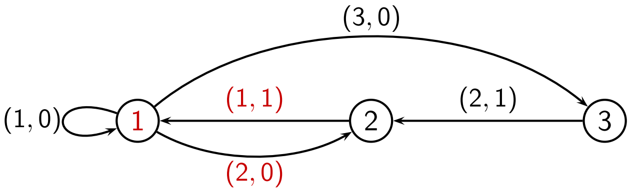

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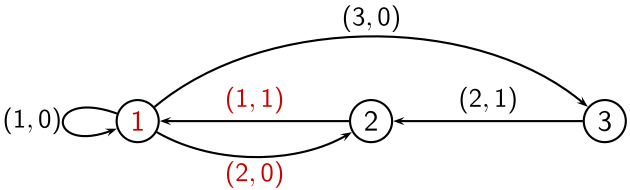
									$k=1$								
$w^{(2)} =$									1								



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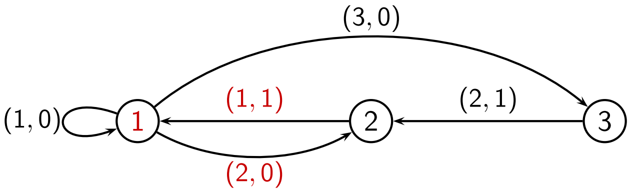

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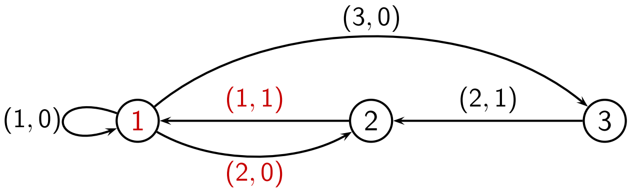

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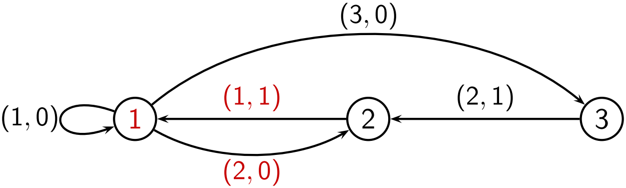

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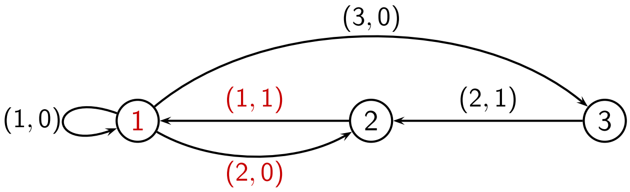
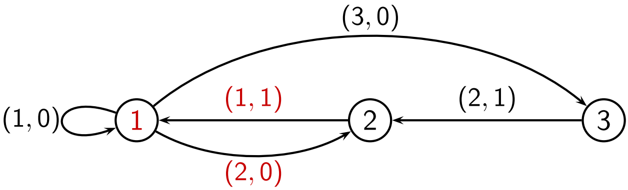
$w^{(1)} =$	···	2	1	1	2	1	3	1	2	1	1	2	1	3	1	2	···
									$k = 1$								
$w^{(2)} =$	···	1	1	2	1	3	1	2	1	3	1	2	1	1	2	1	···



$\Gamma(w_0) = (2, 0),$

Desubstitution

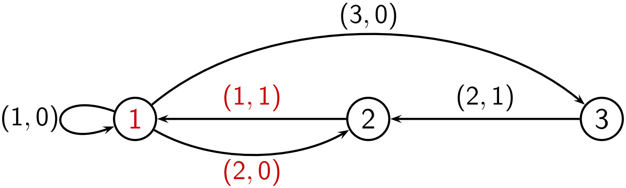
index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	2	1	1	2	1	3	1	2	$\cdots$
									$k = 1$								
$w^{(2)} =$	$\cdots$	1	1	2	1	3	1	2	1	3	1	2	1	1	2	1	$\cdots$



$$\Gamma(w_0) = (2, 0), \quad (1, 1),$$

Desubstitution

index	$\cdots$	$\overset{-}{7}$	$\overset{-}{6}$	$\overset{-}{5}$	$\overset{-}{4}$	$\overset{-}{3}$	$\overset{-}{2}$	$\overset{-}{1}$	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	2	1	1	2	1	3	1	2	$\cdots$
									$k = 1$								
$w^{(2)} =$	$\cdots$	1	1	2	1	3	1	2	1	3	1	2	1	1	2	1	$\cdots$
$\vdots$									$\vdots$								



$$\Gamma(w_0) = (2, 0), \quad (1, 1), \quad \ldots$$

Shift of finite type

We see: Γ maps the elements of Ω onto the shift of finite type induced by the graph.

Shift of finite type

We see: Γ maps the elements of Ω onto the shift of finite type induced by the graph.

Theorem (Canterini-Siegel (2001), Holton-Zamboni (2001))

The map Γ is continuous, surjective and, up to a countable set of possible exceptions, injective. The shift S is conjugate to an adic transformation on the edges (Vershik-map).

Scenario

For $w = (w_j)_{j\in\mathbb{Z}} \in \Omega$ let us

- treat w symmetrically with respect to the division point . (that is, the pair $w_{-1}w_0$ is the "centre" of w) and

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$. 0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	$\cdots$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 .	1	3	1	2	1	2	1	3	$\cdots$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$. 0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$								
$w^{(1)} =$								1 . 2								

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$										
$w^{(1)} =$							3	1	.	2	1							

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index	⋯	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$. 0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	⋯
$w^{(0)} =$	⋯	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	⋯
								$k = 0$								
$w^{(1)} =$	⋯	2	1	1	2	1	3	1 . 2	1	1	2	1	3	1	2	⋯

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$. 0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 . 2	1	1	2	1	3	1	2	$\cdots$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 .	1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 .	2	1	1	2	1	3	1	2	$\cdots$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$

| 2 . 1 |

$(k = 1)$

Scenario

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$
							1	2	.	1	3							$(k = 1)$

Scenario

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$
									$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$
		$\cdots$		2	1	3	1	2	.	1	3	1	2	1	$\cdots$			$(k = 1)$

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 .	1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 .	2	1	1	2	1	3	1	2	$\cdots$
		$\cdots$		2	1	3	1	2 .	1	3	1	2	1		$\cdots$		$(k = 1)$

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								$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 . 2	1	1	2	1	3	1	2	$\cdots$

$\cdots$ 2 1 3 1 | 2 . 1 | 3 1 2 1 $\cdots$ $(k = 1)$

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	$\cdots$
								$k = 0$								
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 . 2	1	1	2	1	3	1	2	$\cdots$

$\cdots$	2	1	3	1	2 . 1	3	1	2	1	$\cdots$		$(k = 1)$
					1 . 3							$(k = -1)$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$. 0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$	
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2 . 1	3	1	2	1	2	1	3	$\cdots$	
								$k = 0$									
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1 . 2	1	1	2	1	3	1	2	$\cdots$	
				$\cdots$	2	1	3	1	2 . 1	3	1	2	1	$\cdots$			$(k = 1)$
				$\cdots$				2	1 . 3	1							$(k = -1)$

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$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$	
									$k = 0$										
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$	
		$\cdots$		2	1	3	1	2	.	1	3	1	2	1	$\cdots$				$(k = 1)$
		$\cdots$		1	3	1	2	1	.	3	1	2	1	1	$\cdots$				$(k = -1)$

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index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$	
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$	
								$k = 0$											
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$	
$w^{(2)} =$?			$\cdots$	2	1	3	1	2	.	1	3	1	2	1		$\cdots$			$(k = 1)$
			$\cdots$	1	3	1	2	1	.	3	1	2	1	1		$\cdots$			$(k = -1)$

Question:

What can we do to obtain uniqueness?

Coding prescriptions

Definition

Let c be a function that assigns to each $a \in \mathcal{A}$ a set $c(a) \subset \mathbb{Z}$ that is a complete set of representatives modulo $|\sigma(a)|$ whose absolute values are smaller than $|\sigma(a)|$. (Thus, $|c(a)| = |\sigma(a)|$ and for each $k \in c(A)$ we have $-|\sigma(a)| < k < |\sigma(a)|$; $k' \in c(a)$ with $k' \neq k$ implies $|\sigma(a)| \not| (k - k')$.) Extend c to $\mathcal{A}^2$ by defining

$$c(ab) = \{k \in c(a) : k \leq 0\} \cup \{k \in c(b) : k \geq 0\}.$$

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Theorem (S. (2016))

Let (Ω, S) be the substitution dynamical system induced by σ, c a coding prescription, and $w \in \Omega$. Then there exists a uniquely determined $w' = (w'_j)_{j \in \mathbb{Z}} \in \Omega$ and a uniquely determined integer $k \in c(w'_{-1} w'_0)$ such that

$$w = S^k \sigma(w').$$

Coding graphs

A coding prescription induces in a natural way a graph G_c with set of vertices $\mathfrak{L}_2 := \mathfrak{L} \cap \mathcal{A}^2$ and an edge from $a'b'$ to ab if there is a $k \in c(ab)$ such that $a'b'$ appears at the $(|\sigma(a)| + k)$th position in $\sigma(ab)$. We label it by (ab, k).

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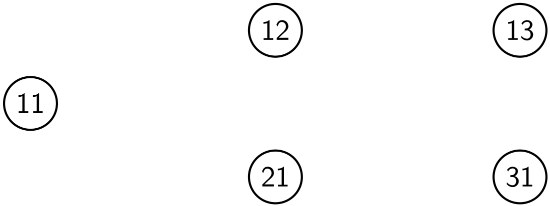
$c : 1 \mapsto \{0, 1\}, 2 \mapsto \{-1, 0\}, 3 \mapsto \{0\}$.
The set of vertices is $\mathfrak{L}_2 = \{11, 12, 13, 21, 31\}$.



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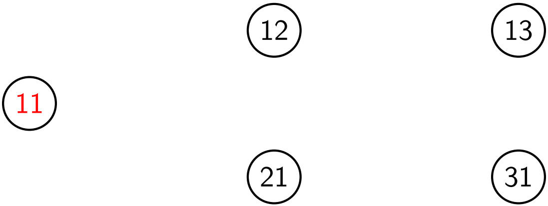
$$\sigma(11) = \ 1 \overset{-1}{\ } 2 \, . \overset{0}{\ } 1 \overset{1}{\ } 2$$



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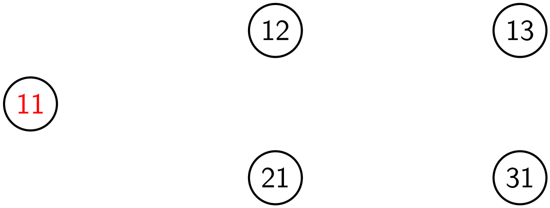
$$\sigma(11) = 1 \overset{-1}{} 2 \overset{0}{.} 1 \overset{1}{} 2$$

$c(11) = \{0, 1\}$



Coding graphs

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$$\sigma(11) = 1 \overset{-1}{} 2 \overset{0}{.} 1 \overset{1}{} 2$$

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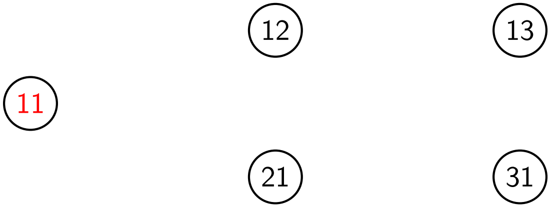

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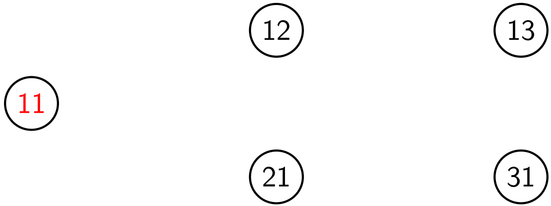

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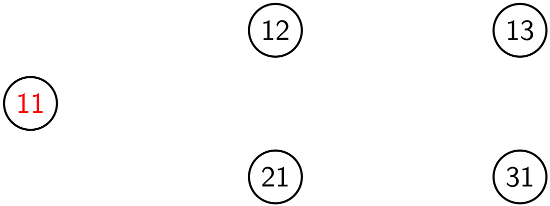

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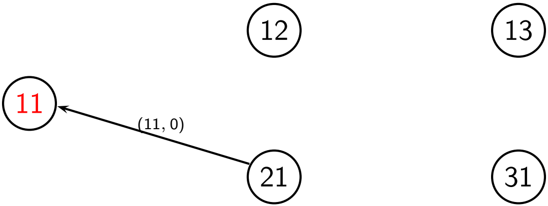
$$\begin{array}{lcccc} & & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} \\ \sigma(11) = & 1 & 2 & 1 & 2 \end{array}$$

$c(11) = \{0, 1\}$



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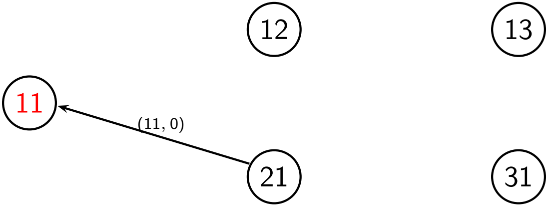

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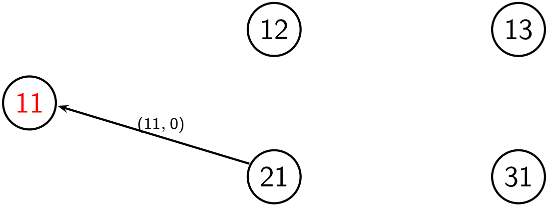

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$$\sigma(11) = 1 \quad \overset{-1}{2} \quad \overset{0}{1} \quad \overset{1}{2}$$

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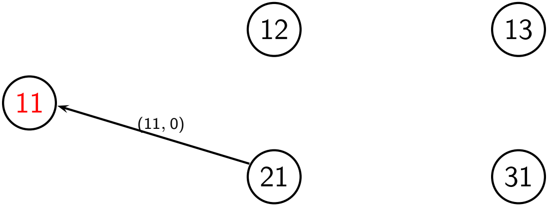

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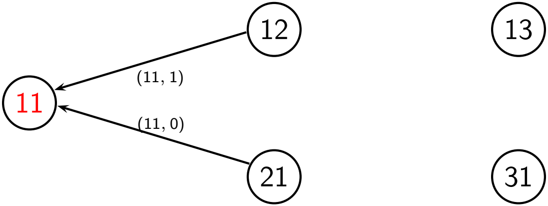

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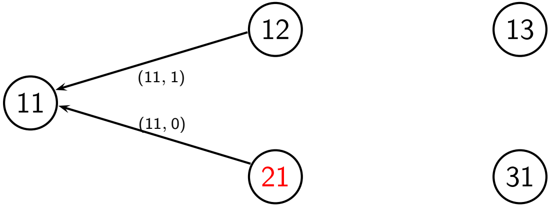
$\sigma(11) = 1\ \overset{-1}{2}\,.\,\overset{0}{1}\ \overset{1}{2}$ $\qquad$ $\sigma(21) = 1\ \overset{-1}{3}\,.\,\overset{0}{1}\ \overset{1}{2}$

$c(11) = \{0, 1\}$



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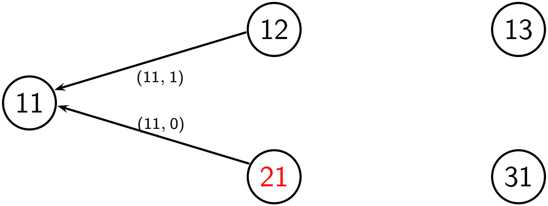

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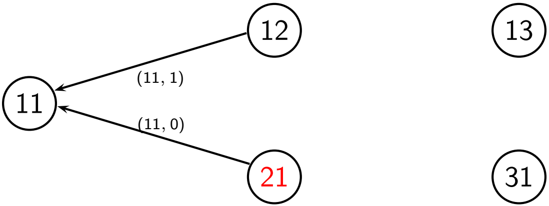

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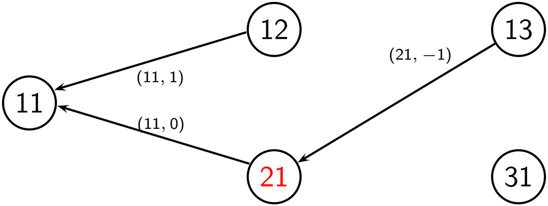

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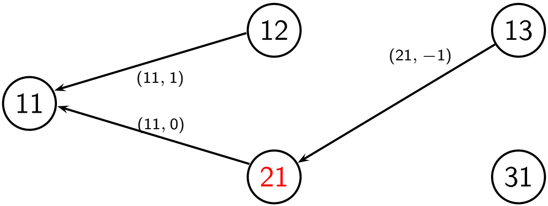

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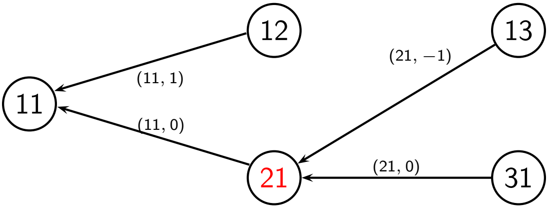

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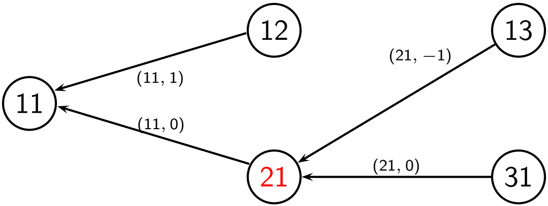

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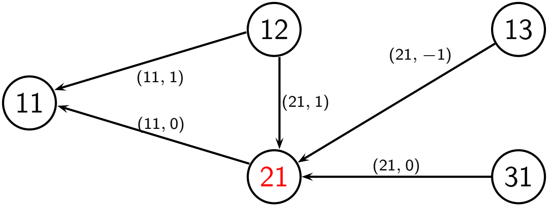
$c(21) = \{-1, 0, 1\}$



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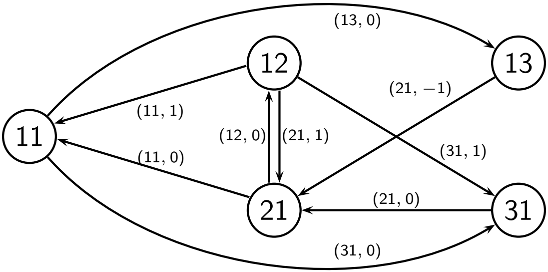
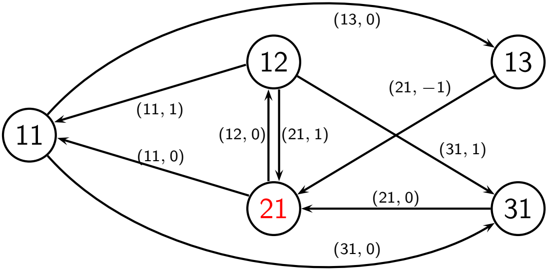


Back to our scenario

index	⋯	$\overset{-}{7}$	$\overset{-}{6}$	$\overset{-}{5}$	$\overset{-}{4}$	$\overset{-}{3}$	$\overset{-}{2}$	$\overset{-}{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	⋯
$w^{(0)} =$	⋯	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	⋯



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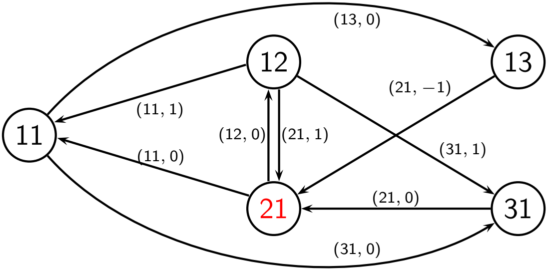
index	$\cdots$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\cdots$
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$
$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$



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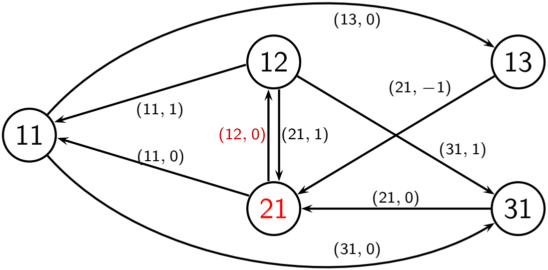
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$

$$k = 0 \in c(12) = \{0\}$$

$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$



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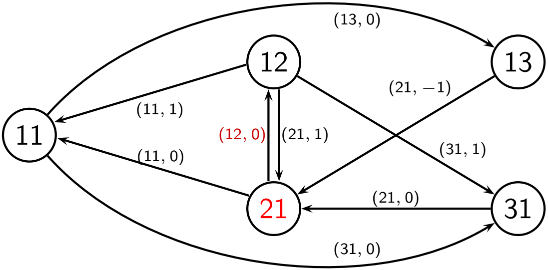
$w^{(0)} =$	$\cdots$	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	$\cdots$

$$k = 0 \in c(12) = \{0\}$$

$w^{(1)} =$	$\cdots$	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	$\cdots$

$w^{(2)} = \{$	$\cdots$	2	1	3	1	2	.	1	3	1	2	1	$\cdots$	$(k = 1)$
	$\cdots$	1	3	1	2	1	.	3	1	2	1	1	$\cdots$	$(k = -1)$



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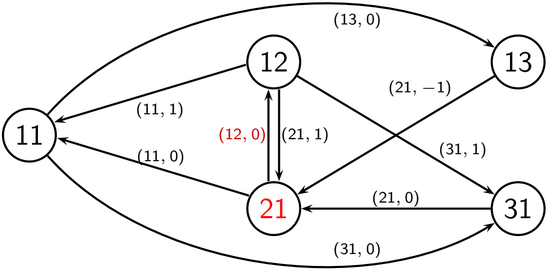
$w^{(0)} =$	···	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	···	
$k = 0 \in c(12) = \{0\}$																			
$w^{(1)} =$	···	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	···	
$c(21) = \{-1, 0, 1\}$																			
$w^{(2)} = \{$			···	2	1	3	1	2	.	1	3	1	2	1	···				$(k = 1)$
			···	1	3	1	2	1	.	3	1	2	1	1	···				$(k = -1)$



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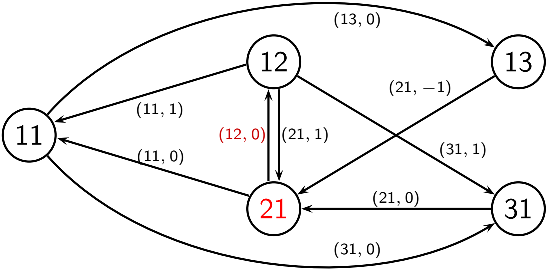
$w^{(0)} =$	⋯	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	⋯	
$k = 0 \in c(12) = \{0\}$																			
$w^{(1)} =$	⋯	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	⋯	
$c(13) = \{0\}$																			
$w^{(2)} =$			⋯	2	1	3	1	2	.	1	3	1	2	1	⋯				$(k = 1)$
			⋯	1	3	1	2	1	.	3	1	2	1	1	⋯				$(k = -1)$



Back to our scenario

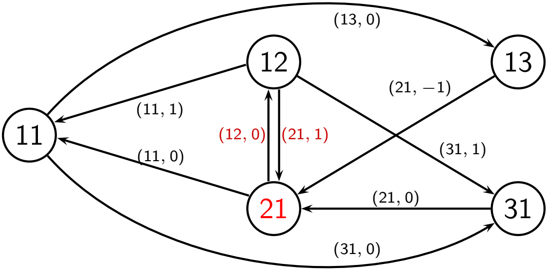
index	…	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	.	0	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	…
$w^{(0)} =$	…	1	3	1	2	1	1	2	.	1	3	1	2	1	2	1	3	…

$$k = 0 \in c(12) = \{0\}$$

$w^{(1)} =$	…	2	1	1	2	1	3	1	.	2	1	1	2	1	3	1	2	…

$$k = 1 \in c(21) = \{-1, 0, 1\}$$

$w^{(2)} =$	…	2	1	3	1	2	.	1	3	1	2	1	…	$(k = 1)$
	…	1	3	1	2	1	.	3	1	2	1	1	…	$(k = -1)$



Shifts of finite type

Let $(w^{(n)})_{n\geq 1} \in \Omega^{\mathbb{N}}$ and $(k_n)_{n\geq 1} \in \mathbb{Z}^{\mathbb{N}}$ with $k_n \in c(w_{-1}^{(n)} w_0^{(n)})$ for all $n \geq 1$ (where $w_{-1}^{(n)} w_0^{(n)}$ denotes central pair of $w^{(n)}$) such that $w^{(n-1)} = S^{k_n}\sigma(w^{(n)})$. Define

$$\Gamma_c(w) := (w_{-1}^{(n)} w_0^{(n)}, k_n) \in (\mathfrak{L}^2 \times \mathbb{Z})^{\mathbb{N}}.$$

The image $\Gamma_c(w)$ corresponds to an infinite walk on G_c. Hence, Γ_c maps Ω onto the shift of finite type induced by G_c.

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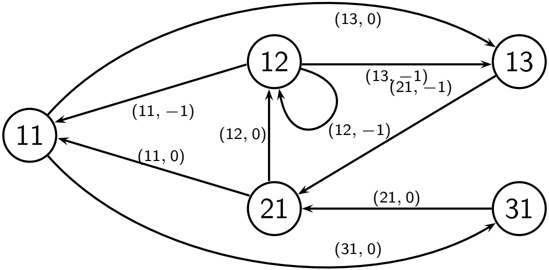
Theorem (S. (2016))

The map Γ_c is continuous, surjective and, up to a countable set of possible exceptions, injective.

Comparison

For each substitution σ there are exactly $\prod_{a\in\mathcal{A}} 2^{|\sigma(a)|-1}$ different coding prescriptions.

$$\begin{array}{llll} \sigma: & 1 \longmapsto 12, & 2 \longmapsto 13 & 3 \longmapsto 1 \\ c: & 1 \longmapsto \{-1,0\}, & 2 \longmapsto \{-1,0\} & 3 \longmapsto \{0\} \end{array}$$



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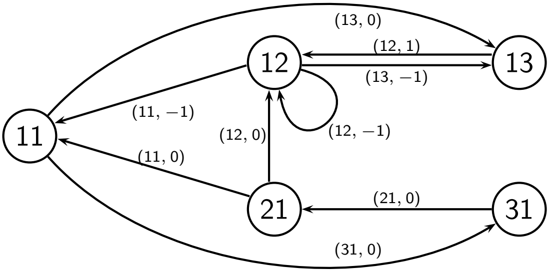

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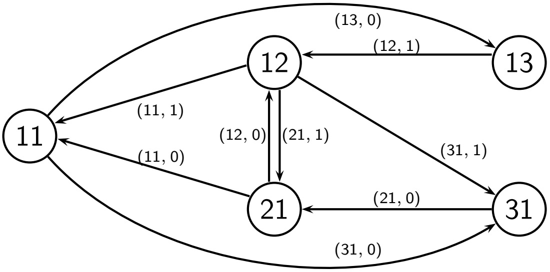

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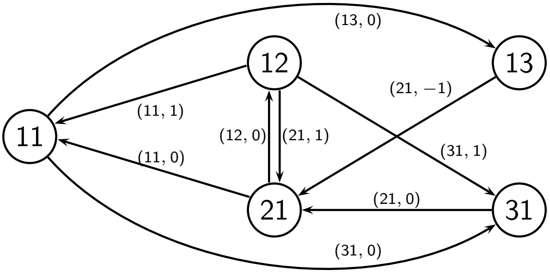

Observations and basic properties

- The shape and even the number of edges depend on the choice of c. The only edges that do not depend on c are those of the shape $(ab, 0)$.

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- By considering higher powers of σ we obtain arbitrary many different ways to code the dynamical system Ω as shift of finite type.
- We always can find a coding prescription such that Γ_c is completely injective (possibly by considering higher powers of σ).
- Choose c such that $c(a)$ contains only non-negative integers for all $a \in \mathcal{A}$ and $w \in \Omega$. If $\Gamma_c(w) = (a_n b_n, k_n)_{n\geq 1}$ then $\Gamma(w) = (b_n, k_n)_{n\geq 1}$. Thus, the coding mentioned at the beginning corresponds to one special choice of c.

Continuous coding prescriptions

We call a coding prescription *continuous* if $c(a)$ consists of consecutive integers for each $a \in \mathcal{A}$.

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Theorem (S. (2016))

If c is continuous then Γ_c conjugates the shift map S with an adic transformation on the edges.

Application: Numeration

Currently I work on an application of the theory in numeration. For a given primitive substitution we can associate with each coding prescription a number system. The coding prescription that assigns to each letter a set of non-negative integers yields the well-known Dumont-Thomas numeration.

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Thank you for your attention