SUBSTITUTIONS, CODING PRESCRIPTIONS AND NUMERATION

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Denote by $\mathcal{A}$ a finite set (alphabet) and by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$. For a word $X \in \mathcal{A}^*$ we write $|X|$ for the length of X. Consider a substitution σ, *i.e.* a morphism $\sigma : \mathcal{A} \to \mathcal{A}^*$ and its extension to $\mathcal{A}^*$ via concatenation.

A *coding prescription (with respect to* σ*)* is a function c with domain $\mathcal{A}^2$ that assigns to each pair of letters a finite set of integers with the following properties:

(1) for all $k \in c(xx)$ we have $|k| < |\sigma(x)|$ (for all $x \in \mathcal{A}$);
(2) $c(xx)$ is a complete set of representatives modulo $|\sigma(x)|$ (for all $x \in \mathcal{A}$);
(3) $c(ab) = \{k \in c(aa) : k \leq 0\} \cup \{k \in c(bb) : k \geq 0\}$ (for all $ab \in \mathcal{A}^2$).

The notion of coding prescription was introduced in [4] in order to code substitution dynamical systems as shifts of finite type. In the actual presentation we want to concentrate on combinatorial aspects of coding prescriptions.

In particular, we will show how to compose coding prescriptions with respect to given substitutions σ and σ' over the same alphabet in order to obtain a coding prescription for the composition $\sigma' \circ \sigma$ or for powers σ^n. This will yield a way to represent integers quite analogously as the Dumont-Thomas numeration for natural integers (see [2, 3]). The set of representable numbers may consist of all (positive and negative) integers, it may have gaps, and it can consist of 0 only. This depends on the actual coding prescription. For a special one, that assigns to each $ab \in \mathcal{A}^2$ a set of non-negative integers, we retrieve exactly the results from [3].

For primitive substitutions we can also base a numeration system for real numbers on our setting where the domain can have various characteristics. Again, this depends on the coding prescription. For the special choice from above we obtain the Dumont-Thomas numeration for real numbers. Other choices of σ and c yield, for example, symmetric beta-expansions (*cf* [1]). We will outline several effects by examples.

References

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[4] P. Surer, *Coding of substitution dynamical systems as shifts of finite type.*, Ergodic Theory Dyn. Syst., 36 (2016), pp. 944–972.