Substitutions, coding prescriptions and Numeration

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Preliminaries

We let $\mathcal{A} := \{1, 2, \ldots, m\}$ be a finite set (alphabet) and denote by $\mathcal{A}^*$ the free monoid over $\mathcal{A}$, ε the empty word and $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$ the set of non-empty words.

We consider a non-erasing morphism (substitution) $\sigma : \mathcal{A}^* \longmapsto \mathcal{A}^*$. We call σ primitive if there exists a positive integer n such that y appears in $\sigma^n(x)$ for all $x, y \in \mathcal{A}$.

Coding prescriptions

Coding Prescription

A *coding prescription (with respect to σ)* is a function c with domain $\mathcal{A}^2$ that assigns to each pair of letters a finite set of integers such that

1. $c(xx)$ is a complete set of representatives modulo $|\sigma(x)|$ for all $x \in \mathcal{A}$, that is $\#c(xx) = |\sigma(x)|$ and for all $k, k' \in c(xx)$ with $k \neq k'$ we have $k \not\equiv k' \mod |\sigma(x)|$;
2. for all $x \in \mathcal{A}$ we have $-|\sigma(x)| < k < |\sigma(x)|$ for all $k \in c(xx)$;
3. for all $ab \in \mathcal{A}^2$ we have $c(ab) = \{k \in c(aa) : k \leq 0\} \cup \{k \in c(bb) : k \geq 0\}$.

We call a coding prescription *continuous* if $c(ab)$ consists of consecutive integers for each $ab \in \mathcal{A}^2$

Some formalism

We define:

- the set of "inverse letters" $\overline{\mathcal{A}} := \{\bar{1}, \bar{2}, \ldots, \bar{m}\}$;
- the equivalence relation $\sim$ on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ induced by the cancellation law $x\bar{x} \sim \varepsilon \sim \bar{x}x$;
- $|X|_y$ for $X \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$, $y \in \mathcal{A}$ to be the difference of the number of occurrences of y and the number of occurrences of $\bar{y}$ in X;
- for $X \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$

$$|X| := \sum_{y \in \mathcal{A}} |X|_y, \qquad \mathbf{l}(X) := (|X|_1, |X|_2, \ldots, |X|_m).$$

Observe that we have compatibility with the concatenation of words and $\sim$: for all $X, X' \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$, $y \in \mathcal{A}$ we have $|XX'|_y = |X|_y + |X'|_y$ and $X \sim X'$ implies $|X|_y = |X'|_y$.

More formalism

For a word $X = x_1 \cdots x_n \in (\mathcal{A} \cup \overline{\mathcal{A}})^*$ we define

$$\overline{X} := \overline{x_n}\,\overline{x_{n-1}} \cdots \overline{x_1}$$

where $\overline{\overline{x}} = x$ for all $x \in \mathcal{A}$.

We extend σ to $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ by defining $\sigma(\bar{x}) = \overline{\sigma(x)}$ for all $x \in \mathcal{A}$.

We introduce the partial ordering $\prec$ on $(\mathcal{A} \cup \overline{\mathcal{A}})^*$ defined by

$$X \prec Y \iff \exists S \in \mathcal{A}^+ : XS \sim Y.$$

The graph associated to a coding prescription

For a substitution σ over the alphabet $\mathcal{A}$ and a coding prescription c with respect to σ we define the directed graph $H_{\sigma,c}$ in the following way.

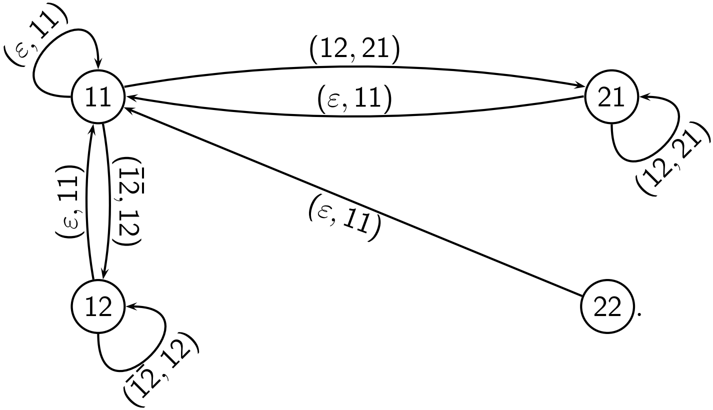
The set of vertices is given by the set $\mathcal{A}^2$.

There is an edge from ab to $a_1 b_1$ labelled by $(D, a_1 b_1) \in \mathcal{A}^* \cup \overline{\mathcal{A}}^* \times \mathcal{A}^2$ if and only if $|D| \in c(ab)$ and there exist words $P, S \in \mathcal{A}^*$ such that $\sigma(a)D \sim Pa_1$ as well as $\overline{D}\sigma(b) \sim b_1 S$.

Example

Let $\mathcal{A} = \{1, 2\}$ and $\sigma : 1 \mapsto 121, 2 \mapsto 1$.

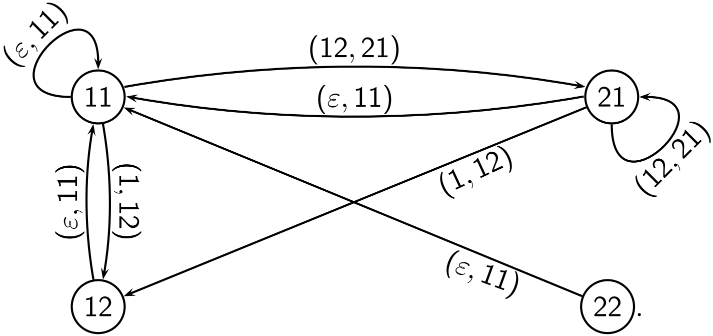
$c_1(11) = \{-2, 0, 2\}, c_1(22) = \{0\}, c_1(12) = \{-2, 0\}, c_1(21) = \{0, 2\}$



Example

Let $\mathcal{A} = \{1, 2\}$ and $\sigma : 1 \mapsto 121, 2 \mapsto 1$.

$$c_2(11) = \{0, 1, 2\}, c_2(22) = \{0\}, c_2(12) = \{0\}, c_2(21) = \{0, 1, 2\}$$



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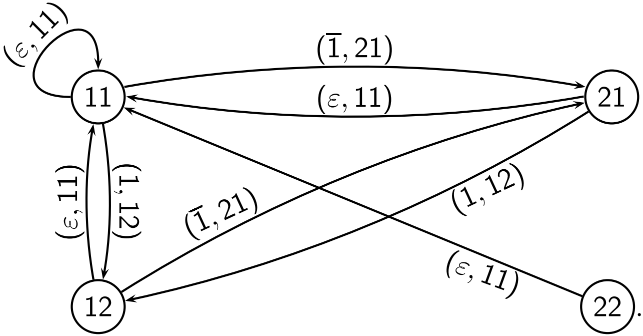
Example

Let $\mathcal{A} = \{1, 2\}$ and $\sigma : 1 \mapsto 121, 2 \mapsto 1$.

$c_3(11) = \{-1, 0, 1\}, c_3(22) = \{0\}, c_3(12) = \{-1, 0\}, c_3(21) = \{0, 1\}$



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Paths in $H_{\sigma,c}$

We denote by $H^n_{\sigma,c}(ab)$ the paths of length n that start in ab where n is either a positive integer or ∞.

For two edges $(D, a_1 b_1)$ and $(D', a'_1 b'_1)$ we write $(D, a_1 b_1) \prec (D', a'_1 b'_1)$ if $D \prec D'$.

Observation

For each $ab \in \mathcal{A}^2$ all outgoing edges are $\prec$-comparable. Furthermore, all outgoing (finite and infinite) paths are $\prec_{\mathrm{lex}}$ comparable.

Composition of coding prescriptions

Let σ_1, σ_2 be two substitutions over the same alphabet $\mathcal{A}$ and c_1 and c_2 coding prescriptions with respect to σ_1 and σ_2, respectively. Denote $H_1 := H_{\sigma_1,c_1}$ and $H_2 := H_{\sigma_2,c_2}$.
For $ab \in \mathcal{A}^2$ define

$$c_2 \odot c_1(ab) := \{|\sigma_2(D_1)D_2| : (D_1, a_1b_1) \in H^1_{\sigma_1,c_1}(ab), \\ (D_2, a_2b_2) \in H^1_{\sigma_2,c_2}(a_1b_1)\}.$$

Theorem

The map $c_2 \odot c_1$ is a coding prescription with respect to $\sigma_2 \circ \sigma_1$.

Continuity

If c_1 and c_2 are both continuous then $c_2 \odot c_1$ is continuous.

Example

Let $\sigma_1 = \sigma_2 = \sigma : 1 \mapsto 121, 2 \mapsto 1$,

$$c_1 : 11 \mapsto \{-2, 0, 2\}, 22 \mapsto \{0\}, 12 \mapsto \{-2, 0\}, 21 \mapsto \{0, 2\},$$
$$c_2 : 11 \mapsto \{0, 1, 2\}, 22 \mapsto \{0\}, 12 \mapsto \{0\}, 21 \mapsto \{0, 1, 2\}$$

$c_2 \odot c_1(11)$:

$H_{\sigma,c_1}(11)$	$H_{\sigma,c_2}(a_1 b_1)$	$\sigma(D_1)D_2$	$\lvert\sigma(D_1)D_2\rvert$
$(\bar{1}\bar{2}, 12)$	$(\varepsilon, 11)$	$\bar{1}\bar{2}\bar{1}\bar{1}$	-4
$(\varepsilon, 11)$	$(\varepsilon, 11)$	ε	0
$(\varepsilon, 11)$	$(1, 12)$	1	1
$(\varepsilon, 11)$	$(12, 21)$	12	2
$(12, 21)$	$(\varepsilon, 11)$	1211	4
$(12, 21)$	$(1, 12)$	12111	5
$(12, 21)$	$(12, 21)$	121112	6

Therefore: $c_2 \odot c_1(11) = \{-4, 0, 1, 2, 4, 5, 6\}$

Furthermore: $c_2 \odot c_1(22) = \{0, 1, 2\}$,

$c_2 \odot c_1(12) = \{-4, 0, 1, 2\}$, $c_2 \odot c_1(21) = \{0, 1, 2, 4, 5, 6\}$

Powers of coding prescriptions

Consider a substitution σ over the alphabet $\mathcal{A}$ and a coding prescriptions c. For an $n \geq 1$ and $ab \in \mathcal{A}^2$ define

$$c^{(n)}(ab) := \left\{ \sum_{j=1}^{n} |\sigma^{n-j}(D_j)| : (D_j, a_j b_j) \in H^n_{\sigma,c}(ab) \right\}.$$

Theorem

For each $n \geq 1$ the map $c^{(n)}$ is a coding prescription with respect to σ^n.

Continuity

$c^{(n)}$ is continuous if and only if c is continuous.

Representation of integers

Suppose that for a pair $ab \in \mathcal{A}^2$ we have $(\varepsilon, ab) \in H^1_{\sigma,c}(ab)$. Then we have for all $n \geq 1$

$$c^{(n)}(ab) \subset c^{(n+1)}(ab).$$

We can use this setting to (uniquely) represent the integers of the set

$$Z_{ab} := \bigcup_{n \in \mathbb{N}} c^{(n)}(ab).$$

Theorem

Each for each $N \in Z_{ab}$ can be represented as

$$N = \sum_{j=1}^{n} |\sigma^{n-j}(D_j)|$$

where $(D_j, a_j b_j)_{1 \leq j \leq n} \in H^n_{\sigma,c}(ab)$ for some $n \geq 0$. The representation is unique wehen we require that $D_1 \neq \varepsilon$ (and 0 is represented by the empty sum).

Properties

The shape of Z_{ab}

- $0 \in Z_{ab}$
- Z_{ab} contains positive (negative, resp.) integers if and only if $c(ab)$ contains at least one positive (negative, resp.) integer.
- Z_{ab} contains all positive (negative, resp.) integers if and only if $1 \in c(ab)$ ($-1 \in c(ab)$, resp.).

Ordering

Let $(D_j, a_j b_j)_{1 \leq j \leq n}, (D_j, a_j b_j)_{1 \leq j \leq n} \in H^n(ab)$ and

$$N = \sum_{j=1}^{n} |\sigma^{n-j}(D_j)|, \quad N' = \sum_{j=1}^{n} |\sigma^{n-j}(D'_j)|.$$

If c is continuous then $N < N'$ if and only if $(D_j, a_j b_j)_{1 \leq j \leq n} \prec_{lex} (D'_j, a'_j b'_j)_{1 \leq j \leq n}$.

Examples

$\sigma : 1 \mapsto 121, 2 \mapsto 1$ over $\mathcal{A} = \{1, 2\}$

- $c_1 : 11 \mapsto \{-2, 0, 2\}, 22 \mapsto \{0\}$:

Z_{11}	path representing 8
$2\mathbb{Z}$	$(12, 21), (\varepsilon, 11), (\bar{1}\bar{2}, 12)$ $(\sigma^2(12)\sigma(\varepsilon)\bar{1}\bar{2} \sim 12111211)$

- $c_2 : 11 \mapsto \{0, 1, 2\}, 22 \mapsto \{0\}$

Z_{11}	path representing 8
$\{0, 1, 2, \ldots\}$	$(1, 12), (\varepsilon, 11), (1, 12)$ $(\sigma^2(1)\sigma(\varepsilon)1 \sim 12111211)$

- $c_3 : 11 \mapsto \{-1, 0, 1\}, 22 \mapsto \{0\}$

Z_{11}	path representing 8
$\mathbb{Z}$	$(1, 12), (\varepsilon, 11), (1, 12)$

Another example

$\sigma^2 : 1 \mapsto 1211121, 2 \mapsto 121$ over $\mathcal{A} = \{1, 2\}$

$c_2 \odot c_1 : 11 \mapsto \{-4, 0, 1, 2, 4, 5, 6\}, 22 \mapsto \{0, 1, 2\}$:

- $Z_{11} = \{0, 1, 2, \ldots\} \cup \{-4, -18, -19, -20, -22, -23, -24, -28, -100, \ldots\}$
- path representing 6: $(12, 12), (\bar{1}\bar{2}\bar{1}\bar{1}, 11)$
- path representing 8: $(1, 11), (1, 12)$

Infinite walks in $H_{\sigma,c}$

We now suppose σ to be primitive. Define

- $\mathbf{M} := (|\sigma(y)|_x)_{1 \leq x,y \leq m}$ the incidence matrix of σ;
- $\theta > 1$ the dominant eigenvalue of $\mathbf{M}$;
- $\mathbf{v} = (v_1, \ldots, v_m)$ a strictly positive left eigenvector of $\mathbf{M}$ with respect to θ;
- $\lambda : (\mathcal{A} \cup \overline{\mathcal{A}})^* \longrightarrow \mathbb{R}, X \longmapsto \langle \mathbf{l}(X), \mathbf{v} \rangle$.
- for a walk $(D_j, a_j b_j)_{j \geq 1} \in H^{\infty}_{\sigma,c}(ab)$

$$\Lambda((D_j)_{j\geq 1}) := \sum_{j\geq 1} \theta^{-j} \lambda(D_j);$$

- each $ab \in \mathcal{A}^2$

$$I_{ab} := \{\Lambda((D_j)_{j\geq 1}) : (D_j, a_j b_j)_{j\geq 1} \in H^{\infty}_{\sigma,c}(ab)\}.$$

The continuous case

For continuous c the following items hold

- There exist vectors $(v_1^+, \dots, v_m^+) \in \mathbb{R}^m$ with non-negative entries and $(v_1^-, \dots, v_m^-) \in \mathbb{R}^m$ with non-positive entries such that $(v_1^+, \dots, v_m^+) - (v_1^-, \dots, v_m^-) = \mathbf{v}$ and for each $ab \in \mathcal{A}^2$

$$I_{ab} = [v_a^-, v_b^+];$$

- for all $ab \in \mathcal{A}^2$ we have

$$I_{ab} = \bigcup_{(D_1, a_1 b_1) \in H^1(ab)} \theta^{-1}(\lambda(D_1) + I_{a_1 b_1}),$$

  where the particular sets have disjoint interior;
- for $(D_j, a_j b_j)_{j \geq 1}, (D'_j, a'_j b'_j)_{j \geq 1} \in H^\infty_{\sigma,c}(ab)$ we have

$$(D_j, a_j b_j)_{j \geq 1} \prec_{\text{lex}} (D'_j, a'_j b'_j)_{j \geq 1} \Longrightarrow \Lambda((D_j)_{j \geq 1}) \leq \Lambda((D'_j)_{j \geq 1}).$$

Numeration

Theorem

Fix $ab \in \mathcal{A}^2$. Then for each $\gamma \in [v_a^-, v_b^+)$ there exists a uniquely determined walk $(D_j, a_j b_j)_{j \geq 1} \in H_{\sigma,c}^{\infty}(ab)$ such that

$$\gamma = \Lambda((D_j)_{j\geq 1}) = \sum_{j \geq 1} \theta^{-j} \lambda(D_j)$$

with the additional condition that $(D_j, a_j b_j)_{j \geq n+1}$ is not the maximal walk starting in $a_n b_n$ for all $n \geq 1$.

Remark

The theorem analogously holds for $(v_a^-, v_b^+]$ and non-minimal walks.

Examples

$\sigma : 1 \mapsto 121, 2 \mapsto 1$ over $\mathcal{A} = \{1, 2\}$

- $c_2 : 11 \mapsto \{0, 1, 2\}, 22 \mapsto \{0\}$:

(v_1^-, v_2^-)	(v_1^+, v_2^+)	Set of digits	Remark
$(0, 0)$	(v_1, v_2)	$\{0, \lambda(1), \lambda(12)\}$	Dumont-Thomas numeration

- $c_3 : 11 \mapsto \{-1, 0, 1\}, 22 \mapsto \{0\}$

(v_1^-, v_2^-)	(v_1^+, v_2^+)	Digits	Remark
$(-\frac{v_1}{2}, -\frac{v_2}{2})$	$(\frac{v_1}{2}, \frac{v_2}{2})$	$\{\lambda(\bar{1}), 0, \lambda(1)\}$	for $\mathbf{v} = (1, \theta^{-1})$: symmetric beta-expansion w.r.t. $\theta = 1 + \sqrt{2}$

A special case

Suppose that for all $x \in \mathcal{A}$ we have $|\sigma(x)| \equiv 0 \mod 2$ and let c be the (unique) coding prescription with $c(ab) \subset 2\mathbb{Z}$ for all $ab \in \mathcal{A}^2$. Then the following items hold.

- $I_{ab} = [-v_a, v_b]$ for each $ab \in \mathcal{A}^2$;
- for all $ab \in \mathcal{A}^2$ we have

$$I_{ab} = \bigcup_{(D_1, a_1 b_1) \in H^1(ab)} \theta^{-1}(\lambda(D_1) + I_{a_1 b_1}),$$

  where the particular sets have disjoint interior;
- for $(D_j, a_j b_j)_{j \geq 1}, (D'_j, a'_j b'_j)_{j \geq 1} \in H^{\infty}_{\sigma,c}(ab)$ we have

$$(D_j, a_j b_j)_{j \geq 1} \prec_{\mathrm{lex}} (D'_j, a'_j b'_j)_{j \geq 1} \Longrightarrow \Lambda((D_j)_{j \geq 1}) \leq \Lambda((D'_j)_{j \geq 1}).$$

Numeration

Consider the setting from above.

Theorem

Fix $ab \in \mathcal{A}^2$. *Then for each* $\gamma \in [-v_a, v_b)$ *there exists a uniquely determined walk* $(D_j, a_j b_j)_{j\geq 1} \in H^{\infty}_{\sigma,c}(ab)$ *such that*

$$\gamma = \Lambda((D_j)_{j\geq 1}) = \sum_{j\geq 1} \theta^{-j} \lambda(D_j)$$

with the additional condition that $(D_j, a_j b_j)_{j\geq n+1}$ *is not the maximal walk starting in* $a_n b_n$ *for all* $n \geq 1$.

Example

$\sigma : 1 \mapsto 121, 2 \mapsto 1$ over $\mathcal{A} = \{1, 2\}$

$c_1 : 11 \mapsto \{-2, 0, 2\}, 22 \mapsto \{0\}$:

I_{11}	I_{22}	Digits
$[-v_1, v_1]$	$[-v_2, v_2]$	$\{\lambda(\bar{1}\bar{2}), 0, \lambda(12)\}$

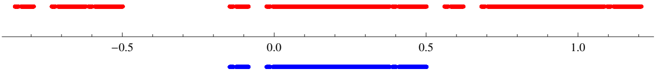
For $\mathbf{v} = (\theta(\theta+1)^{-1}, (\theta+1)^{-1}) = (\sqrt{2}/2, 1 - \sqrt{2}/2)$ we obtain integer digits.

Example

For $\sigma^2 : 1 \longmapsto 1211121, 2 \mapsto 121$ and

$$c = c_2 \odot c_1 : 11 \mapsto \{-4, 0, 1, 2, 4, 5, 6\}, 22 \mapsto \{0, 1, 2\}$$

the set I_{11} is depicted in red and I_{22} is depicted in blue.



Thank You for your attention