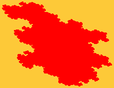

Fractal tiles associated to generalised radix representations and shift radix systems

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Shift Radix Systems

Definition (*cf.* Akiyama *et al.*, 2005)

Let $\mathbf{r} \in \mathbb{R}^d$ and

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \mathbf{x} = (x_1, \ldots, x_d) \to (x_2, \ldots, x_d, -\lfloor \mathbf{r}\mathbf{x} \rfloor).$$

The dynamical system $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ is called a shift radix system (SRS). The SRS satisfies the finiteness property if

$$\forall \mathbf{x} \in \mathbb{Z}^d : \exists n \in \mathbb{N} \text{ such that } \tau_{\mathbf{r}}^n(\mathbf{x}) = \mathbf{0}.$$

Notations

Notation

- For $\mathbf{r} = (r_0, \ldots, r_{d-1})$ denote by $M_{\mathbf{r}}$ the companion matrix with characteristic polynomial $\chi_{M_{\mathbf{r}}}(x) = x^d + r_{d-1}x^{d-1} + \cdots + r_0$.
- $\mathcal{E}_d := \{\mathbf{r} \in \mathbb{R}^d | \varrho(M_{\mathbf{r}}) < 1\}$.

Proposition

$\mathbf{r} \in \mathcal{E}_d$ the SRS $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ either satisfies the finiteness property or for all $\mathbf{x} \in \mathbb{Z}^d$ the sequence $(\tau_{\mathbf{r}}^n(\mathbf{x}))_{n\in\mathbb{N}}$ is ultimately periodic..

SRS-tiles

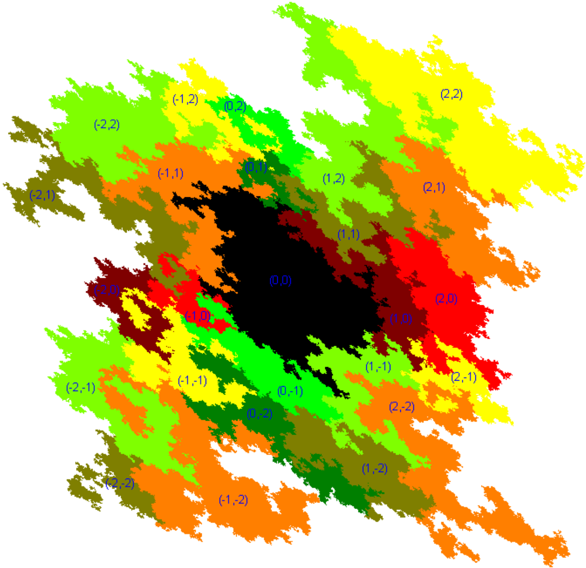
Definition

Let $\mathbf{r} \in \mathcal{E}_d$ and $\mathbf{x} \in \mathbb{Z}^d$. The set

$$\mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \lim_{n \to \infty} M_{\mathbf{r}}^n \tau_{\mathbf{r}}^{-n}(\mathbf{x})$$

(limit with respect to the Hausdorff metric) is called the SRS tile associated with $\mathbf{r}$. $\mathcal{T}_{\mathbf{r}}(\mathbf{0})$ is called the central SRS tile associated with $\mathbf{r}$.

SRS-tiles for $\mathbf{r} = (\frac{3}{4}, 1)$



Basic properties of SRS tiles

For each $\mathbf{r} \in \mathcal{E}_d$ we have

- $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ is compact for all $\mathbf{x} \in \mathbb{Z}^d$.
- The family $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) | \mathbf{x} \in \mathbb{Z}^d\}$ is locally finite.
- $$\bigcup_{\mathbf{x} \in \mathbb{Z}^d} \mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \mathbb{R}^d.$$
- $\mathcal{T}_{\mathbf{r}}(\mathbf{x})$ satisfies the set equation

$$\mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \bigcup_{\mathbf{y} \in \tau_{\mathbf{r}}^{-1}(\mathbf{x})} M_{\mathbf{r}} \mathcal{T}_{\mathbf{r}}(\mathbf{y}).$$

Periodic points

Definition

For $\mathbf{r} \in \mathbb{R}^d$ a point $\mathbf{z} \in \mathbb{Z}^d$ is called purely periodic (with respect to $\tau_{\mathbf{r}}$) if $\tau_{\mathbf{r}}^l(\mathbf{z}) = \mathbf{z}$ for some $l \geq 1$.

Proposition

For each $\mathbf{r} \in \mathcal{E}_d$ there exists only finitely many purely periodic points. $\mathbf{0}$ is the only purely periodic point if and only if $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ has the finiteness property.

SRS tiles and the origin

Let $\mathbf{r} \in \mathcal{E}_d$.

- $\mathbf{0} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})$ if and only if $\mathbf{x}$ is purely periodic.
- $\tau_{\mathbf{r}}$ is an SRS if and only if $\mathbf{0} \in \mathcal{T}_{\mathbf{r}}(\mathbf{0}) \setminus \bigcup_{\mathbf{x} \neq 0} \mathcal{T}_{\mathbf{r}}(\mathbf{x})$ is an inner point of the central tile.

Closure of the interior

Note

SRS tiles are not necessarily the closure of the interior!

Example

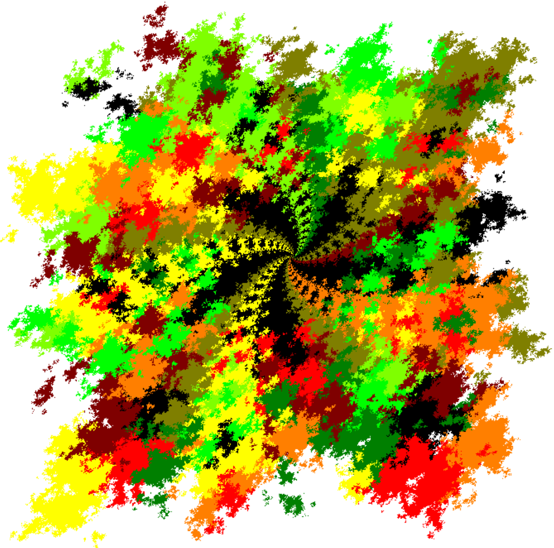
Set $\mathbf{r} = (\frac{9}{10}, -\frac{11}{20})$. The points $\mathbf{z}_0 = (-1,-1), \mathbf{z}_1 = (-1,1), \mathbf{z}_2 = (1,2), \mathbf{z}_3 = (2,1), \mathbf{z}_4 = (1,-1)$ are purely periodic:

$$\tau_{\mathbf{r}} : \mathbf{z}_0 \mapsto \mathbf{z}_1 \mapsto \mathbf{z}_2 \mapsto \mathbf{z}_3 \mapsto \mathbf{z}_4 \mapsto \mathbf{z}_0.$$

But $\tau_{\mathbf{r}}^{-n}(\mathbf{z}_0) = \{\mathbf{z}_{(n \mod 5)}\}$ and thus

$$\mathcal{T}_{\mathbf{r}}(\mathbf{z}_0) = \mathcal{T}_{\mathbf{r}}(\mathbf{z}_1) = \mathcal{T}_{\mathbf{r}}(\mathbf{z}_2) = \mathcal{T}_{\mathbf{r}}(\mathbf{z}_3) = \mathcal{T}_{\mathbf{r}}(\mathbf{z}_4) = \{\mathbf{0}\}.$$

SRS tiles for $\mathbf{r} = \left(\frac{9}{10}, -\frac{11}{20}\right)$ (Modern Art)



Tiling properties

Definition

Let $\mathbf{r} \in \mathcal{E}_d$. The family $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) | \mathbf{x} \in \mathbb{Z}^d\}$ provides a weak *m*-tiling if for $m+1$ pairwise different points $\mathbf{x}_1, \ldots, \mathbf{x}_{m+1}$ we have $\bigcap_{i=1}^{m+1} \operatorname{int}(\mathcal{T}_{\mathbf{r}}(\mathbf{x})) = \emptyset$ and for all points $t \in \mathbb{R}^d$ we have $\#\{\mathbf{x} \in \mathbb{Z}^d | \mathbf{t} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})\} \geq m$. We call a point $t \in \mathbb{R}^d$ an *m*-exclusive point if $\#\{\mathbf{x} \in \mathbb{Z}^d | \mathbf{t} \in \mathcal{T}_{\mathbf{r}}(\mathbf{x})\} = m$.

Note

- SRS tiles are not necessarily the closure of the interior.
- The family $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) | \mathbf{x} \in \mathbb{Z}^d\}$ is not necessarily a collection of finitely many tiles up to translation (Counterexample: $\mathbf{r} = (-\frac{2}{3})$).
- We are not able to prove in general that the boundaries of the SRS tiles have zero measure.

Weak m-tiling

Theorem

Let $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathcal{E}_d$. *The family* $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) | \mathbf{x} \in \mathbb{Z}^d\}$ *provides a weak m-tiling if one of the following conditions hold.*

- $\mathbf{r} \in \mathbb{Q}^d$,
- $r_0, \ldots, r_{d-1}$ *are algebraically independent over* $\mathbb{Q}$,
- $(x - \beta)(x^d + r_{d-1}x^{d-1} + \cdots + r_0) \in \mathbb{Z}[x]$ *for some* $\beta > 1$.

Corollary

Let $\mathbf{r} \in \mathcal{E}_d$. *If* $\mathbf{r}$ *satisfies one of the conditions from above and the SRS* $(\mathbb{Z}^d, \tau_{\mathbf{r}})$ *satisfies the finiteness property then the family* $\{\mathcal{T}_{\mathbf{r}}(\mathbf{x}) | \mathbf{x} \in \mathbb{Z}^d\}$ *provides a weak (1-)tiling.*

Tiles associated to an expanding polynomial

Definition (*cf.* Kátai, Kőrnyei)

Let $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ an expanding polynomial ($\Rightarrow |a_0| \geq 2$) and B the transposed companion matrix with characteristic polynomial A.

$$\mathcal{F} := \left\{ \mathbf{t} \in \mathbb{R}^d \,\middle|\, \mathbf{t} = \sum_{i=0}^{\infty} B^{-i}(c_i, 0, \ldots, 0)^T, c_i \in \mathcal{N} \right\}$$

($\mathcal{N} = \{0, \ldots, |a_0| - 1\}$) is called self-affine tile associated with A.

Lemma

- *$\mathcal{F}$ is compact and self-affine.*
- *$\mathcal{F}$ is the closure of its interior.*
- $\{\mathbf{x} + \mathcal{F}, \mathbf{x} \in \mathbb{Z}^d\}$ *defines a tiling of* $\mathbb{R}^d$.

Relation to SRS-tiles

$$\mathbf{r} = \left(\frac{1}{a_0}, \frac{a_{d-1}}{a_0}, \dots, \frac{a_1}{a_0}\right), \quad V = \begin{pmatrix} 1 & a_{d-1} & \cdots & a_1 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{d-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Theorem

For all $\mathbf{x} \in \mathbb{Z}^d$ *we have*

$$\begin{aligned} \mathcal{F} &= V T_{\mathbf{r}}(\mathbf{0}), \\ \mathbf{x} + F &= V T_{\mathbf{r}}(V^{-1}(\mathbf{x})). \end{aligned}$$

Example: $A(x) = x^2 - x + 3$

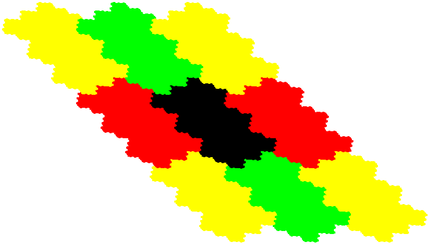


Figure: Translates of the self-affine tile associated with A

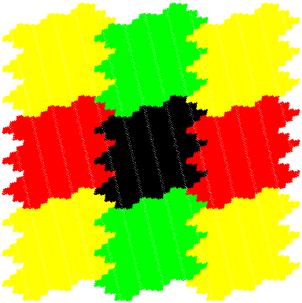


Figure: SRS tile associated with $\left(\frac{1}{3}, -\frac{1}{3}\right)$

Tiles associated to Pisot number

Setting

Let $\beta > 1$ a Pisot number with minimal Polynomial $(x-\beta)(x^d + r_{d-1}x^{d-1} + \cdots + r_0)$,

$$T_\beta : \mathbb{Z}[\beta] \cap [0,1) \longrightarrow \mathbb{Z}[\beta] \cap [0,1), \gamma \mapsto \beta\gamma - \lfloor \beta\gamma \rfloor,$$

$\beta = \beta_0, \beta_1, \dots, \beta_d$ the galois conjugates of β, $d = p + 2q$,
$\beta_0, \dots, \beta_p \in \mathbb{R}$,
$\beta_{p+1} = \overline{\beta_{p+1+q}}, \dots, \beta_{p+q} = \overline{\beta_{p+2q}} \in \mathbb{C}$,
$\gamma^{(i)}$ the corresponding conjugate of $\gamma \in \mathbb{Q}(\beta)$, $i \in \{0, \dots, d\}$,

$$\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^d, \gamma \mapsto \left(\gamma^{(1)}, \dots, \gamma^{(p+q)}\right).$$

Tiles associated to Pisot numbers

Theorem (Akiyama *et al.*)

$(\mathbb{Z}^d, \tau_{\mathbf{r}})$ has the finiteness property if and only if β has the property (F).

Definition (*cf.* Akiyama)

For $\omega \in \mathbb{Z}[\beta] \cap [0,1)$ the set

$$\mathcal{S}_\beta(\omega) = \lim_{n\to\infty} \Phi(\beta^n T_\beta^{-n}(\omega))$$

(with the Hausdorff limit) is called integral β-tile.

Lemma

For units we have finitely many tiles up to translation. Each tile is the closure of its interior.

Relation between SRS-tiles and integral β-tiles

Let
$$f : \mathbb{Z}^d \to \mathbb{Z}[\beta] \cap [0,1), \mathbf{x} \mapsto \mathbf{r}\mathbf{x} - \lfloor \mathbf{r}\mathbf{x} \rfloor$$
(Bijective map!)

Theorem

There exists a matrix U such that for each $\mathbf{x} \in \mathbb{Z}^d$ we have that $\mathcal{S}_\beta(f(\mathbf{x})) = UT_{\mathbf{r}}(\mathbf{x})$.

Corollary

Let β a Pisot number of degree $d+1$ satisfying the property (F). Then the family $\{\mathcal{S}_\beta(\omega)\}_{\omega \in \mathbb{Z}[\beta] \cap [0,1)}$ is a weak tiling of $\mathbb{R}^d$.

Example (Pisot unit case)

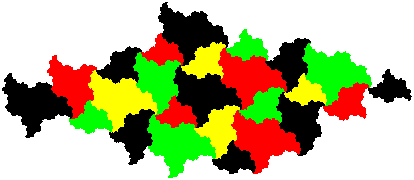


Figure: Integral beta-tiles for β the smallest Pisot number

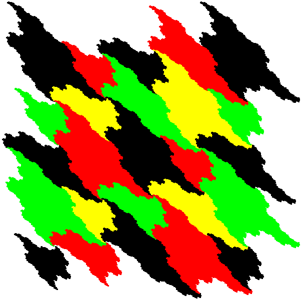


Figure: The corresponding SRS tiles

Example (Pisot non-unit case)

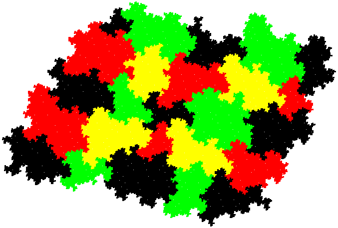


Figure: Integral beta-tiles for β with minimal polynomial $x^3 - 3x^2 - x - 2$.

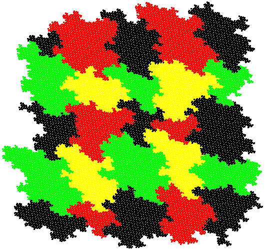


Figure: The corresponding SRS tiles

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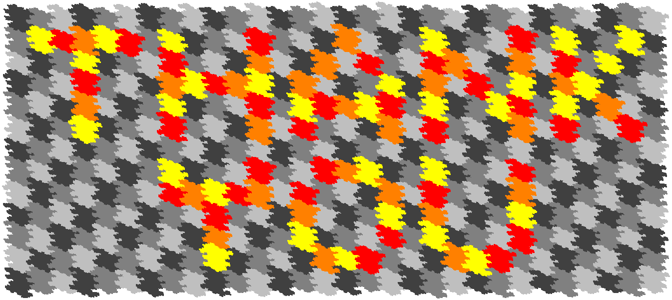

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